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Mathematical Problems in Mechanics

Justification of the Darboux–Vallée–Fortuné compatibility relation in the theory of surfaces

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Abstract

Given two fields of positive definite symmetric, and symmetric, matrices defined over a simply-connected open subset $\omega \subset \mathbb{R}^2$, the fundamental theorem of surface theory asserts that, if these fields satisfy the Gauss and Codazzi–Mainardi relations in ω, then there exists an immersion $\boldsymbol{\theta}$ from ω into $\mathbb{R}^3$ such that these fields are the two fundamental forms of the surface $\boldsymbol{\theta}(\omega)$

We show here that a new compatibility relation, shown to be necessary by C. Vallée and D. Fortuné in 1996 through the introduction, following an idea of G. Darboux, of a rotation field on a surface, is also sufficient for the existence of such an immersion $\boldsymbol{\theta}$. ***To cite this article: P.G. Ciarlet, O. Iosifescu, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Justification de la condition de compatibilité de Darboux–Vallée–Fortuné en théorie des surfaces. Etant donné deux champs suffisamment réguliers définis dans un ouvert simplement connexe $\omega \subset \mathbb{R}^2$, l'un de matrices symétriques définies positives et l'autre de matrices symétriques, le théorème fondamental de la théorie des surfaces affirme que, si ces deux champs satisfont les relations de Gauss et Codazzi–Mainardi dans ω, alors il existe une immersion $\boldsymbol{\theta}$ de ω dans $\mathbb{R}^3$ telle que ces champs soient les deux formes fondamentales de la surface $\boldsymbol{\theta}(\omega)$.

On montre ici qu'une nouvelle relation de compatibilité, dont C. Vallée et D. Fortuné ont montré en 1996 la nécessité en suivant une idée de G. Darboux, est également suffisante pour l'existence d'une telle immersion $\boldsymbol{\theta}$. ***Pour citer cet article : P.G. Ciarlet, O. Iosifescu, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

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1. Introduction

Let ω be an open subset of $\mathbb{R}^2$ and let $\boldsymbol{\theta} \in \mathcal{C}^3(\omega; \mathbb{R}^3)$ be an immersion. The *first* and *second fundamental forms* $(a_{\alpha\beta}) \in \mathcal{C}^2(\omega; \mathbb{S}^2_>)$ and $(b_{\alpha\beta}) \in \mathcal{C}^1(\omega; \mathbb{S}^2_>)$ of the *surface* $\boldsymbol{\theta}(\omega) \subset \mathbb{R}^3$ are then defined by means of their covariant components

$$a_{\alpha\beta} := \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{\theta} \quad \text{and} \quad b_{\alpha\beta} := \partial_{\alpha\beta} \boldsymbol{\theta} \cdot \frac{\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}}{|\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}|}.$$

The matrix fields $(a_{\alpha\beta})$ and $(\beta_{\alpha\beta})$ cannot be arbitrary: Let $C_{\alpha\beta\tau} := \frac{1}{2}(\partial_\beta a_{\alpha\tau} + \partial_\alpha a_{\beta\tau} - \partial_\tau a_{\alpha\beta})$ and $C^\sigma_{\alpha\beta} := a^{\sigma\tau} C_{\alpha\beta\tau}$, where $(a^{\sigma\tau}) := (a_{\alpha\beta})^{-1}$. Then the *Gauss* and *Codazzi–Mainardi compatibility relations*, viz.

$$\partial_\beta C_{\alpha\sigma\tau} - \partial_\sigma C_{\alpha\beta\tau} + C^\nu_{\alpha\beta} C_{\sigma\tau\nu} - C^\nu_{\alpha\sigma} C_{\beta\tau\nu} = b_{\alpha\sigma} b_{\beta\tau} - b_{\alpha\beta} b_{\sigma\tau}, \tag{1}$$

$$\partial_\beta b_{\alpha\sigma} - \partial_\sigma b_{\alpha\beta} + C^\nu_{\alpha\sigma} b_{\beta\nu} - C^\nu_{\alpha\beta} b_{\sigma\nu} = 0, \tag{2}$$

necessarily hold in ω.

When ω is *simply-connected*, the relations (1)–(2) become also *sufficient* for the existence of such a mapping $\boldsymbol{\theta}$, according to the following classical *fundamental theorem of surface theory*: Let $\omega \subset \mathbb{R}^2$ be open and simply-connected and let $(a_{\alpha\beta}) \in \mathcal{C}^2(\omega; \mathbb{S}^2_>)$ and $(b_{\alpha\beta}) \in \mathcal{C}^1(\omega; \mathbb{S}^2)$ satisfy the *Gauss and Codazzi–Mainardi compatibility relations* in ω. *Then there exists an immersion $\boldsymbol{\theta} \in \mathcal{C}^3(\omega; \mathbb{R}^3)$ such that $(a_{\alpha\beta})$ and $(b_{\alpha\beta})$ are the first and second fundamental forms of the surface $\boldsymbol{\theta}(\omega)$.*

The purpose of this Note is to show that the Gauss and Codazzi–Mainardi relations can be replaced by a *different*, more "geometrical" and substantially simpler, *necessary compatibility relation* that has been obtained in *vector form*, through the introduction of an appropriate *rotation field on a surface*, by Vallée and Fortuné [11], following an idea that goes back to Darboux [3].

It is to be emphasized that the most striking feature of the *Darboux–Vallée–Fortuné compatibility relation* is its *geometrical nature*, illustrated by its relation to a *surface rotation field*. That, by contrast with the Gauss and Codazzi–Mainardi equations, the Christoffel symbols do *not* appear in the Darboux–Vallée–Fortuné relation is equally noteworthy.

Particularly relevant to the present work are the interesting analyses of Pietraszkiewicz and Vallée [8], Pietraszkiewicz and Szwabowicz [6], and Pietraszkiewicz, Szwabowicz and Vallée [7], who showed how the midsurface of a deformed thin shell can be reconstructed from the knowledge of the undeformed midsurface and of the surface strains and bendings. In particular, these authors also use in a crucial way the polar factorization of the deformation gradient of the midsurface.

2. Notations

Latin and Greek indices range respectively in $\{1, 2, 3\}$ and $\{1, 2\}$ and the summation convention with respect to repeated indices is used. The symbols $\mathbb{M}^n$, $\mathbb{M}^{m\times n}$, $\mathbb{S}^n$, and $\mathbb{S}^n_>$ designate the sets of all $n \times n$, $m \times n$, $n \times n$ symmetric, and $n \times n$ positive definite symmetric, real matrices.

Given any matrix $\mathbf{A} \in \mathbb{M}^n$, its *cofactor matrix* (also a matrix in $\mathbb{M}^n$) is denoted $\mathbf{COF\,A}$. Given any matrix $\mathbf{C} \in \mathbb{S}^n_>$, there exists a unique matrix $\mathbf{U} \in \mathbb{S}^n_>$ such that $\mathbf{U}^2 = \mathbf{C}$. The matrix $\mathbf{U}$ is denoted $\mathbf{C}^{1/2}$.

The coordinates of a point $x \in \mathbb{R}^3$ are denoted x_i and partial derivatives operators of the first order are devoted ∂_i. The coordinates of a point $y \in \mathbb{R}^2$ are denoted y_α and partial derivatives of the first and second order are denoted ∂_α and $\partial_{\alpha\beta}$.

Given a mapping $\boldsymbol{\Theta} = (\Theta_i) \in \mathcal{D}'(\Omega; \mathbb{R}^3)$, the matrix field $\boldsymbol{\nabla\Theta} \in \mathcal{D}'(\Omega; \mathbb{M}^3)$ is defined by $(\boldsymbol{\nabla\Theta})_{ij} = \partial_j \Theta_i$. Given a matrix field $\mathbf{A} = (a_{ij}) \in \mathcal{D}'(\Omega; \mathbb{M}^3)$, the matrix field $\mathbf{CURL\,A} \in \mathcal{D}'(\Omega; \mathbb{M}^3)$ is that whose i-th row vector is the rotational of the i-th row vector of $\mathbf{A}$.

3. A new formulation of the fundamental theorem of Riemannian geometry in $\mathbb{R}^3$

The fundamental theorem of Riemannian geometry in $\mathbb{R}^3$ classically asserts that, if the Riemann curvature tensor associated with a field $\mathbf{C} \in \mathcal{C}^1(\Omega; \mathbb{S}^3_>)$ vanishes in a simply-connected open subset Ω of $\mathbb{R}^3$, then $\mathbf{C}$ is the *metric* *tensor field* of a manifold isometrically embedded in $\mathbb{R}^3$, i.e., there exists an immersion $\boldsymbol{\Theta} \in \mathcal{C}^2(\Omega;\mathbb{R}^3)$ such that $\mathbf{C} = \boldsymbol{\nabla}\boldsymbol{\Theta}^T\boldsymbol{\nabla}\boldsymbol{\Theta}$ in Ω (for a proof under this weakened regularity assumptions, see [4]).

It has recently been shown in [1] that a similar existence theorem holds, but under a *different* compatibility relation, this time involving the *square root of the matrix field* $\mathbf{C}$. More specifically, the following *new formulation of the fundamental theorem of Riemannian geometry in* $\mathbb{R}^3$ has been established in Theorem 6.2 in [1].

Theorem 1. *Let Ω be a simply-connected open subset of $\mathbb{R}^3$ and let $\mathbf{C} \in \mathcal{C}^1(\Omega;\mathbb{S}^3_>)$ be a matrix field that satisfies the Shield–Vallée compatibility relation (so named after Shield [9] and Vallée [10]):*

$$\mathbf{CURL}\,\boldsymbol{\Lambda} + \mathbf{COF}\,\boldsymbol{\Lambda} = \mathbf{0} \quad \text{in } \mathcal{D}'\big(\Omega;\mathbb{M}^3\big), \tag{3}$$

where the matrix field $\boldsymbol{\Lambda} \in \mathcal{C}^0(\Omega;\mathbb{M}^3)$ is defined in terms of the matrix field $\mathbf{C}$ by

$$\boldsymbol{\Lambda} := \frac{1}{\det\widetilde{\mathbf{U}}}\widetilde{\mathbf{U}}\left\{(\mathbf{CURL}\,\widetilde{\mathbf{U}})^T\widetilde{\mathbf{U}} - \frac{1}{2}\big(\mathrm{tr}\big[(\mathbf{CURL}\,\widetilde{\mathbf{U}})^T\widetilde{\mathbf{U}}\big]\big)\mathbf{I}\right\}, \quad \text{where } \widetilde{\mathbf{U}} := \mathbf{C}^{1/2} \in \mathcal{C}^1\big(\Omega;\mathbb{S}^3_>\big). \tag{4}$$

Then there exists an immersion $\boldsymbol{\Theta} \in \mathcal{C}^2(\Omega;\mathbb{R}^3)$ that satisfies

$$\boldsymbol{\nabla}\boldsymbol{\Theta}^T\boldsymbol{\nabla}\boldsymbol{\Theta} = \mathbf{C} \quad \text{in } \mathcal{C}^1\big(\Omega;\mathbb{S}^3_>\big). \tag{5}$$

Such an immersion $\boldsymbol{\Theta}$ becomes uniquely defined if Ω is connected and conditions such as

$$\boldsymbol{\Theta}(x_0) = \mathbf{a}_0 \quad \text{and} \quad \boldsymbol{\nabla}\boldsymbol{\Theta}(x_0) = \mathbf{F}_0 \tag{6}$$

are imposed, where $x_0 \in \Omega$, $a_0 \in \mathbb{R}^3$, and $\mathbf{F}_0 \in \mathbb{M}^3$ is any matrix that satisfies $\mathbf{F}_0^T\mathbf{F}_0 = \mathbf{C}(x_0)$ (for instance, $\mathbf{F}_0 = (\mathbf{C}(x_0))^{1/2}$).

Theorem 1 is the point of departure for our subsequent analysis.

4. A new formulation of the fundamental theorem of surface theory

We now state the main result of this paper, viz., that the *Darboux–Vallée–Fortuné compatibility relation* (cf. (7) below; so named after Darboux [3] and Vallée and Fortuné [11]), which was shown in [11] to be *necessarily* satisfied by the matrix fields $(a_{\alpha\beta}) : \omega \to \mathbb{S}^2_>$ and $(b_{\alpha\beta}) : \omega \to \mathbb{S}^2$ associated with a *given* smooth immersion $\boldsymbol{\theta} : \omega \to \mathbb{R}^3$, are also *sufficient* for the *existence* of such an immersion $\boldsymbol{\theta} : \omega \to \mathbb{R}^3$ if the open set $\omega \subset \mathbb{R}^2$ is *simply-connected*.

Theorem 2. *Let ω be simply-connected open subset of $\mathbb{R}^2$ and let $\mathbf{A} = (a_{\alpha\beta}) \in \mathcal{C}^1(\omega;\mathbb{S}^2_>)$ and $\mathbf{B} = (b_{\alpha\beta}) \in \mathcal{C}^1(\omega;\mathbb{S}^2)$ be two matrix fields that satisfy the Darboux–Vallée–Fortuné compatibility relation*

$$\partial_2\boldsymbol{\lambda}_1 - \partial_1\boldsymbol{\lambda}_2 = \boldsymbol{\lambda}_1 \wedge \boldsymbol{\lambda}_2 \quad \text{in } \mathcal{D}'\big(\omega;\mathbb{R}^3\big), \tag{7}$$

where the components $\lambda_{\alpha\beta} \in \mathcal{C}^1(\omega)$ and $\lambda_{3\beta} \in \mathcal{C}^0(\omega)$ of the two vector fields $\boldsymbol{\lambda}_\beta = (\lambda_{i\beta}) : \omega \to \mathbb{R}^3$ are defined in terms of the matrix fields $\mathbf{A}$ and $\mathbf{B}$ by the matrix equations

$$\begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} := -\mathbf{J}\mathbf{A}^{-1/2}\mathbf{B}, \tag{8}$$

$$(\lambda_{31};\quad \lambda_{32}) := \big(\partial_2 u^0_{11} - \partial_1 u^0_{12};\quad \partial_2 u^0_{21} - \partial_1 u^0_{22}\big)\mathbf{J}\mathbf{A}^{-1/2}\mathbf{J}, \tag{9}$$

where

$$\mathbf{J} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \big(u^0_{\alpha\beta}\big) := \mathbf{A}^{1/2} \in \mathcal{C}^1\big(\omega;\mathbb{S}^2_>\big). \tag{10}$$

Then there exists an immersion $\boldsymbol{\theta} \in \mathcal{C}^2(\omega;\mathbb{R}^3)$ such that

$$\partial_\alpha\boldsymbol{\theta} \cdot \partial_\beta\boldsymbol{\theta} = a_{\alpha\beta} \quad \text{in } \mathcal{C}^1(\omega) \quad \text{and} \quad \partial_{\alpha\beta}\boldsymbol{\theta} \cdot \frac{\partial_1\boldsymbol{\theta} \wedge \partial_2\boldsymbol{\theta}}{|\partial_1\boldsymbol{\theta} \wedge \partial_2\boldsymbol{\theta}|} = b_{\alpha\beta} \quad \text{in } \mathcal{C}^0(\omega). \tag{11}$$

Sketch of proof. We only indicate here, without proofs, the successive steps of the proof, which is otherwise long and technical. Complete details are found in [2].

(i) Let ω_0 be an open subset of $\mathbb{R}^2$ such that $\overline{\omega}_0$ is a compact subset of ω. Then there exists $\varepsilon_0 = \varepsilon_0(\omega_0) > 0$ such that $\{\mathbf{A}(y) - 2x_3\mathbf{B}(y) + x_3^2\mathbf{B}(y)\mathbf{A}^{-1}(y)\mathbf{B}(y)\} \in \mathbb{S}^2_>$ for all $(y, x_3) \in \overline{\Omega}_0$, and $\operatorname{tr}(\mathbf{A}(y)^{1/2} - x_3 \operatorname{tr}(\mathbf{B}(y)\mathbf{A}(y)^{-1/2}) > 0$ for all $(y, x_3) \in \overline{\Omega}_0$, where $\Omega_0 := \omega_0 \times]-\varepsilon_0, \varepsilon_0[$.

To see this, it suffices to combine a straightforward compactness-continuity argument with the assumptions that $\mathbf{A} \in \mathcal{C}^1(\omega; \mathbb{S}^2_>)$ and $\mathbf{B} \in \mathcal{C}^1(\omega; \mathbb{S}^2)$.

(ii) Define the matrix field

$$\mathbf{U} := \left(\mathbf{A} - 2x_3\mathbf{B} + x_3^2\mathbf{B}^{-1}\mathbf{A}\mathbf{B}\right)^{1/2} \in \mathcal{C}^1\left(\overline{\Omega}_0; \mathbb{S}^2_>\right).$$

Then the field $\mathbf{U}$ is also given by

$$\mathbf{U} = \mathbf{Q}^T\left(\mathbf{A}^{1/2} - x_3\mathbf{A}^{-1/2}\mathbf{B}\right),$$

where $(\mathbf{Q})_{11} = (\mathbf{Q})_{22} = \cos\varphi$, $(\mathbf{Q})_{21} = -(\mathbf{Q})_{12} = \sin\varphi$, the function $\varphi \in \mathcal{C}^1(\overline{\Omega}_0)$ being defined by

$$\varphi := \arctan\left(\frac{x_3 \operatorname{tr}(\mathbf{BJA}^{-1/2})}{\operatorname{tr}\mathbf{A}^{1/2} - x_3 \operatorname{tr}(\mathbf{BA}^{-1/2})}\right) \quad \text{with } \mathbf{J} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The matrix field $\mathbf{U} \in \mathcal{C}^1(\overline{\Omega}_0; \mathbb{S}^2_>)$ being defined as above, define the matrix fields

$$\widetilde{\mathbf{U}} := \begin{pmatrix} \boxed{\mathbf{U}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{pmatrix} \in \mathcal{C}^1\left(\overline{\Omega}_0; \mathbb{S}^3_>\right) \quad \text{and} \quad (g_{ij}) := \widetilde{\mathbf{U}}^2 \in \mathcal{C}^1\left(\overline{\Omega}_0; \mathbb{S}^3_>\right).$$

Then $g_{\alpha\beta} = a_{\alpha\beta} - 2x_3b_{\alpha\beta} + x_3^2a^{\sigma\tau}b_{\alpha\sigma}b_{\beta\tau}$ and $g_{i3} = \delta_{i3}$, where $(a^{\sigma\tau}) := (a_{\alpha\beta})^{-1}$.

(iii) The matrix field $\widetilde{\mathbf{U}} \in \mathcal{C}^1(\Omega_0; \mathbb{S}^3_>)$ being defined as in (ii), define the matrix field

$$\boldsymbol{\Lambda} := \frac{1}{\det\widetilde{\mathbf{U}}}\widetilde{\mathbf{U}}\left\{(\mathbf{CURL}\,\widetilde{\mathbf{U}})^T\widetilde{\mathbf{U}} - \frac{1}{2}\left(\operatorname{tr}\left[(\mathbf{CURL}\,\widetilde{\mathbf{U}})^T\widetilde{\mathbf{U}}\right]\right)\mathbf{I}\right\} \in \mathcal{C}^0\left(\overline{\Omega}_0; \mathbb{M}^3\right).$$

Then the field $\boldsymbol{\Lambda}$ is also given by

$$\boldsymbol{\Lambda} = \begin{pmatrix} \boxed{\mathbf{J}^T\mathbf{Q}^T\mathbf{A}^{-1/2}\mathbf{B}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} \Lambda_{31} & \Lambda_{32} \end{matrix} & \partial_3\varphi \end{pmatrix},$$

where the function φ is defined as in (ii), and

$$(\Lambda_{31}; \quad \Lambda_{32}) := (\partial_1u_{12} - \partial_2u_{11}; \quad \partial_1u_{22} - \partial_2u_{21})\mathbf{J}^T\mathbf{U}^{-1}\mathbf{J} \in \mathcal{C}^0\left(\overline{\Omega}_0; \mathbb{M}^{1\times 2}\right), \quad \text{with } (u_{\alpha\beta}) := \mathbf{U}.$$

(iv) The row vector field $(\Lambda_{31}; \Lambda_{32}) \in \mathcal{C}^0(\overline{\Omega}_0; \mathbb{M}^{1\times 2})$ as defined in (iii) is also given by $(\Lambda_{31}; \ \Lambda_{32}) = (\lambda_{31}; \ \lambda_{32}) + (\partial_1\varphi; \ \partial_2\varphi)$, where $(\lambda_{31}; \quad \lambda_{32}) := (\partial_2u^0_{11} - \partial_1u^0_{12}; \ \partial_2u^0_{21} - \partial_1u^0_{22})\mathbf{JA}^{-1/2}\mathbf{J}$ and $(u^0_{\alpha\beta}) := \mathbf{A}^{1/2}$.

The compatibility relation (7) (which has not yet been used so far) plays an indispensable role here.

(v) By parts (iii) and (iv), the matrix field $\boldsymbol{\Lambda} \in \mathcal{C}^0(\overline{\Omega}_0; \mathbb{M}^3)$ defined in (iii) is of the form

$$\boldsymbol{\Lambda} = (\Lambda_{ij}) = \begin{pmatrix} \boxed{\mathbf{J}^T\mathbf{Q}^T\mathbf{A}^{-1/2}\mathbf{B}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} \Lambda_{31} & \Lambda_{32} \end{matrix} & \Lambda_{33} \end{pmatrix} = \left(\begin{array}{c|c|c} \boldsymbol{\Lambda}_1 & \boldsymbol{\Lambda}_2 & \boldsymbol{\Lambda}_3 \end{array}\right), \quad \text{with } \boldsymbol{\Lambda}_3 := \begin{pmatrix} 0 \\ 0 \\ \partial_3\varphi \end{pmatrix},$$

where the row-vector field $(\Lambda_{31}; \Lambda_{32}) \in \mathcal{C}^0(\overline{\Omega}_0; \mathbb{M}^{1\times 2})$ is defined in (iv). Then

$$\mathbf{COF}\,\boldsymbol{\Lambda} = \left(\begin{array}{c|c} \boxed{-(\partial_3\varphi)\mathbf{Q}^T\mathbf{A}^{-1/2}\mathbf{BJ}} & \\ & \boldsymbol{\Lambda}_1 \wedge \boldsymbol{\Lambda}_2 \\ \begin{matrix} 0 & 0 \end{matrix} & \end{array}\right) \in \mathcal{C}^0\left(\overline{\Omega}_0; \mathbb{M}^3\right),$$

and

$$\mathbf{CURL}\,\boldsymbol{\Lambda} = \left(\begin{array}{c|c} \boxed{-\partial_3(\mathbf{J}^T\mathbf{Q}^T\mathbf{A}^{-1/2}\mathbf{B}\mathbf{J})} & \partial_1\boldsymbol{\Lambda}_2 - \partial_2\boldsymbol{\Lambda}_1 \\ 0 \qquad 0 & \end{array}\right) \in \mathcal{D}'\big(\Omega_0; \mathbb{M}^3\big).$$

(vi) Let the matrix fields $\mathbf{COF}\,\boldsymbol{\Lambda} \in \mathcal{C}^0(\overline{\Omega}_0; \mathbb{M}^3)$ and $\mathbf{CURL}\,\boldsymbol{\Lambda} \in \mathcal{D}'(\Omega_0; \mathbb{M}^3)$ be given as in (v). Then

$$\mathbf{CURL}\,\boldsymbol{\Lambda} + \mathbf{COF}\,\boldsymbol{\Lambda} = \mathbf{0} \quad \text{in } \mathcal{D}'\big(\Omega_0; \mathbb{M}^3\big).$$

Like in part (iv), the compatibility relation (7) plays an indispensable role here.

(vii) Let ω be a simply-connected open subset of $\mathbb{R}^2$. Then there exist open subsets ω_n, $n \geqslant 0$, of $\mathbb{R}^2$ such that ω_n is a compact subset of ω for each $n \geqslant 0$ and $\omega = \bigcup_{n\geqslant 0}\omega_n$. Furthermore, for each $n \geqslant 0$, there exists $\varepsilon_n = \varepsilon_n(\omega_n) > 0$ such that the conclusions of part (i) hold with the set $\overline{\Omega}_0$ replaced by $\overline{\Omega}_n$, where $\Omega_n := \omega_n \times]-\varepsilon_n, \varepsilon_n[$. Finally, the open set $\Omega := \bigcup_{n\geqslant 0}\Omega_n$ is connected and simply-connected.

(viii) Let the matrix field $\mathbf{U} \in \mathcal{C}^1(\Omega; \mathbb{S}^2_>)$ be defined by

$$\mathbf{U}(y, x_3) := \big(\mathbf{A}(y) - 2x_3\mathbf{B}(y) + x_3^2\mathbf{B}^{-1}(y)\mathbf{A}(y)\mathbf{B}(y)\big)^{1/2} \in \mathbb{S}^2_> \quad \text{for}(y, x_3) \in \Omega_n,\ n \geqslant 0,$$

and let the matrix field $\boldsymbol{\Lambda} \in \mathcal{C}^0(\Omega; \mathbb{M}^3)$ be defined in terms of the matrix field $\widetilde{\mathbf{U}} \in \mathcal{C}^1(\Omega; \mathbb{S}^3_>)$ (see part (ii)) by Eq. (4). Then

$$\mathbf{CURL}\,\boldsymbol{\Lambda} + \mathbf{COF}\,\boldsymbol{\Lambda} = \mathbf{0} \quad \text{in } \mathcal{D}'\big(\Omega; \mathbb{M}^3\big).$$

(ix) Given any two linearly independent vectors $\mathbf{a}_\alpha^0 \in \mathbb{R}^3$ that satisfy $\mathbf{a}_\alpha^0 \cdot \mathbf{a}_\beta^0 = a_{\alpha\beta}(y_0)$, define the matrix

$$\mathbf{F}_0 := \left(\begin{array}{c|c|c} \mathbf{a}_1^0 & \mathbf{a}_2^0 & \dfrac{\mathbf{a}_1^0 \wedge \mathbf{a}_2^0}{|\mathbf{a}_1^0 \wedge \mathbf{a}_2^0|} \end{array}\right), \quad \text{which satisfies } \mathbf{F}_0^T\mathbf{F}_0 = \begin{pmatrix} \boxed{\mathbf{A}(y_0)} & \begin{matrix} 0 \\ 0 \end{matrix} \\ 0 \quad 0 & 1 \end{pmatrix} = \widetilde{\mathbf{U}}^2(y_0, 0).$$

Since the compatibility relation (3) is satisfied, Theorem 1 shows that there exists a unique immersion $\boldsymbol{\Theta} \in \mathcal{C}^2(\Omega; \mathbb{R}^3)$ that satisfies $\nabla\boldsymbol{\Theta}^T\nabla\boldsymbol{\Theta} = \mathbf{U}^2$ in $\mathcal{C}^1(\Omega; \mathbb{S}^2_>)$, and $\boldsymbol{\Theta}(y_0, 0) = \mathbf{a}_0$ and $\nabla\boldsymbol{\Theta}(y_0, 0) = \mathbf{F}_0$.

Let the mapping $\boldsymbol{\theta} \in \mathcal{C}^2(\omega; \mathbb{R}^3)$ be defined by $\boldsymbol{\theta}(y) := \boldsymbol{\Theta}(y, 0)$ for all $y \in \omega$. Then $\boldsymbol{\theta}$ is an immersion and it satisfies

$$\partial_\alpha\boldsymbol{\theta} \cdot \partial_\beta\boldsymbol{\theta} = a_{\alpha\beta} \quad \text{in } \mathcal{C}^1(\omega), \qquad \partial_{\alpha\beta}\boldsymbol{\theta} \cdot \frac{\partial_1\boldsymbol{\theta} \wedge \partial_2\boldsymbol{\theta}}{|\partial_1\boldsymbol{\theta} \wedge \partial_2\boldsymbol{\theta}|} = b_{\alpha\beta} \quad \text{in } \mathcal{C}^0(\omega),$$

and

$$\boldsymbol{\theta}(y_0) = \mathbf{a}_0 \quad \text{and} \quad \partial_\alpha\boldsymbol{\theta}(y_0) = \mathbf{a}_\alpha^0. \qquad \square$$

Thanks to deep global existence theorems for Pfaff systems with little regularity recently obtained by S. Mardare [5], the existence result of Theorem 2 can be extended to cover the situation where the given fields $(a_{\alpha\beta})$ and $(b_{\alpha\beta})$ are only in the space $W^{1,\infty}_{\mathrm{loc}}(\omega; \mathbb{S}^2_>)$ and $W^{1,\infty}_{\mathrm{loc}}(\omega; \mathbb{S}^2)$, in which case the resulting immersion $\boldsymbol{\theta}$ is found in the space $W^{2,\infty}_{\mathrm{loc}}(\omega; \mathbb{R}^3)$.

By contrast with the *Gauss* and *Codazzi–Mainardi relations* (whose necessity is easy to establish from the knowledge of an immersion), establishing the necessity of the *Darboux–Vallée–Fortuné relation* $\partial_2\boldsymbol{\lambda}_1 - \partial_1\boldsymbol{\lambda}_2 = \boldsymbol{\lambda}_1 \wedge \boldsymbol{\lambda}_2$ through a direct computation turns out to be substantially less easy, however. See Theorem 5.1 in [2].

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