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Numerical Analysis

# Convergence of the finite volume MPFA O scheme for heterogeneous anisotropic diffusion problems on general meshes

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**Abstract**

This Note proves the convergence of the finite volume MultiPoint Flux Approximation (MPFA) O scheme for anisotropic and heterogeneous diffusion problems. Its main originality is that our framework and proof deal with general polygonal and polyhedral meshes as well as with $L^\infty$ diffusion coefficients, which is essential in practical applications. ***To cite this article: L. Agelas, R. Masson, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***



**Résumé**

**Convergence du schéma volume fini multi-point de type "O" pour les problèmes de diffusion hétérogène anisotrope sur maillages généraux.** Cette Note démontre la convergence du schéma volume fini de type « O » pour les problèmes de diffusion en milieu hétérogène anisotrope. Sa principale originalité est de traiter des maillages polygonaux et polyédriques généraux ainsi que des coefficients de diffusion $L^\infty$, ce qui est essentiel dans les applications. ***Pour citer cet article : L. Agelas, R. Masson, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***



## Version française abrégée

Les schémas MPFA de type O (voir [1,6]) sont très utilisés dans l'approximation des flux de Darcy des modèles d'écoulements polyphasiques en milieux poreux car ce sont des schémas volume fini centrés adaptés aux milieux anisotropes et hétérogènes et aux maillages polyédriques généraux. Leur construction repose sur des variables intermédiaires de sous face $u_\sigma^s$ autour de chaque sommet $s$ et pour chaque face $\sigma$, $s \in \sigma$. Ces variables servent à construire un gradient discret consistant $(\overline{\nabla}_{\mathcal{D}} u)_K^s$ dans chaque maille $K$ autour du sommet $s$ où $u$ est le vecteur de l'ensemble des inconnues de sous faces $u_\sigma^s$ et de mailles $u_K$. Les inconnues de sous faces sont ensuite éliminées localement par la conservation des flux sur chaque sous face autour du sommet $s$. Nous introduisons dans cet article la forme variationnelle discrète non symétrique $a_{\mathcal{D}}$ (3) basée sur le gradient discret consistant $(\overline{\nabla}_{\mathcal{D}} u)_K^s$ et le gradient faible $(\widetilde{\nabla}_{\mathcal{D}} u)_K^s$ ainsi que les termes de stabilisation consistants $R_{K,\sigma}^s(u)$. On montre que la formulation variationnelle hy-

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bride associée est équivalente au schéma volume fini hybride (3) et qu'elle généralise le schéma en O. Les inconnues de sous faces sont éliminées par la conservation (4) des flux aux sous faces autour de chaque sommet $s$. Moyennant une hypothèse suffisante et locale de coercivité (5) de la forme variationnelle $a_{\mathcal{D}}$, on montre que le schéma volume fini introduit est bien posé et converge dans $L^2$ vers la solution faible du problème (1) pour des tenseurs de diffusion dans $L^\infty$. On montre de plus la convergence dans $L^2$ du gradient discret vers le gradient de la solution faible (voir le théorème 4.1). Les résultats numériques présentés illustrent comme prédit par l'analyse que le schéma est convergent lorsque la déformation du maillage ou l'anisotropie n'est pas trop forte. Dans le cas contraire, le schéma perd ses propriétés de coercivité et de convergence.

## 1. Introduction

Let $\Omega$ be an open bounded connected polygonal subset of $\mathbb{R}^d$, with $d \in \mathbb{N}^\star$, and $\partial\Omega = \overline{\Omega} \setminus \Omega$ its boundary. In this Note, we consider the elliptic equation on the domain $\Omega$

$$\begin{cases} \operatorname{div}(-\Lambda \nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

with the following hypotheses on the data: $f \in L^2(\Omega)$, $\Lambda$ is in $L^\infty(\Omega, \mathcal{M}_d)$ where $\mathcal{M}_d$ is the set of $d \times d$ symmetric matrices, and there exist $0 < \alpha_0 \leqslant \beta_0$, such that $\mathrm{Sp}(\Lambda(x)) \in [\alpha_0, \beta_0]$ for all $x \in \Omega$ where $\mathrm{Sp}(M)$ stands for the spectrum of the symmetric square matrix $M$.

The MultiPoint Flux Approximation (MPFA) O method is a widely used cell centered finite volume scheme in the oil industry for the discretization of diffusion fluxes in multiphase Darcy porous media flow. The main advantages of this scheme are to provide linear formulae for the fluxes at each face of the mesh in terms of the neighbouring cell unknowns, and to reproduce locally linear solutions on general unstructured meshes. In addition, the MPFA O scheme is adapted to discontinuous, anisotropic diffusion coefficients in the sense that it reproduces cellwise linear solutions for cellwise constant diffusion tensors.

Its construction uses in addition to the cell unknowns $u_K$ for each cell $K$ of the mesh, the intermediate unknowns $u_\sigma^s$ for each face (edge in 2D) $\sigma$ of the mesh and each vertex $s$ of the face $\sigma$. Roughly speaking, assuming that each vertex $s$ of any cell $K$ is shared by exactly $d$ faces $\sigma$ of the cell $K$, subfluxes $F_{K,\sigma}^s$ are built using a cellwise constant diffusion coefficient and a linear approximation of $u$ on the cell $K$ using the cell unknown $u_K$ and the $d$ face unknowns $u_\sigma^s$. Then, the intermediate unknowns are eliminated by the flux continuity equations on each face around the vertex $s$, and the approximate flux $F_{K,\sigma}$ is the sum of the subfluxes over the vertices of the face $\sigma$. A generalization of this construction is proposed in [9] and [4] for general polyhedral meshes.

Recent papers have studied the convergence of the MPFA O scheme but there is yet no convergence result on general polygonal and polyhedral meshes, and none taking into account discontinuous diffusion coefficients which are essential in oil industry applications. In [11,2,10] the convergence of the scheme is obtained on quadrilateral meshes. The proofs are based on equivalences of the MPFA O scheme to mixed finite element methods using specific quadrature rules. To our knowledge, these equivalences do not carry over to general polyhedral meshes in 2D and 3D. In [13] a mimetic finite difference scheme is introduced which is equivalent to the MPFA O scheme for simplicial and parallepipedic cells providing a convergence result for such meshes. This mimetic finite difference scheme is very similar to the schemes proposed in [12] and [5] in 2D. These last three schemes are unconditionally symmetric and coercive but they are not consistent on general polygonal and polyhedral meshes and they have been shown numerically to be non convergent on randomly refined quadrangular grids.

In this paper an hybrid discrete variational formulation is introduced in Section 3 using the framework described in [7,8]. For usual meshes such that each vertex of any cell $K$ is shared by exactly $d$ faces of the cell $K$, our discrete variational formulation is equivalent to the usual MPFA O scheme, provided that the normal vectors to the $d$ faces of each cell $K$ sharing a vertex $s$ of $K$ span $\mathbb{R}^d$. It will in addition provide a generalization of the O scheme on more general polyhedral meshes, alternative to the one described in [9].

In Section 4, a sufficient local condition for the coercivity of the scheme is derived which will yield existence, and uniqueness of the solution. Under this coercivity condition, the convergence of the scheme including the case of $L^\infty$ diffusion coefficients can be proved.

In the following, the weak solution of (1) will be denoted by $\bar{u}$, and $\lambda_{\min}(M)$ denotes the smallest eigenvalue of a given symmetric square matrix $M$.

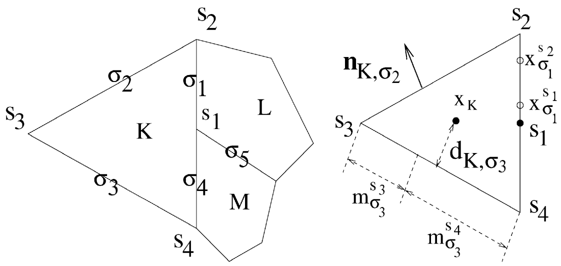


Fig. 1. Example of an admissible mesh in the sense of Definition 2.1 for $d=2$ (left) and examples of notations for the cell $K$ (right). On the left figure, one has for instance $\mathcal{V}_{\sigma_1}=\{s_1,s_2\}$, $\mathcal{V}_K=\{s_1,s_2,s_3,s_4\}$, $\mathcal{E}_{s_1}=\{\sigma_1,\sigma_4,\sigma_5\}$, $\mathcal{E}_K=\{\sigma_1,\sigma_2,\sigma_3,\sigma_4\}$, $\mathcal{T}_{s_1}=\{K,L,M\}$, $\mathcal{T}_{\sigma_1}=\{K,L\}$, $\mathcal{T}_{\sigma_2}=\{K\}$.

## 2. Discrete functional framework

**Definition 2.1.** An admissible finite volume discretization of $\Omega$, denoted by $\mathcal{D}$, is given by $\mathcal{D}=(\mathcal{T},\mathcal{E},\mathcal{P},\mathcal{V})$, where:

- $\mathcal{T}$ is a finite family of non empty connected open disjoint subsets of $\Omega$ (the "cells") such that $\overline{\Omega}=\bigcup_{K\in\mathcal{T}}\overline{K}$. For any $K\in\mathcal{T}$, let $\partial K=\overline{K}\setminus K$ be the boundary of $K$ and $\mathrm{m}_K>0$ denote the measure of $K$.
- $\mathcal{E}$ is a finite family of disjoint subsets of $\overline{\Omega}$ (the "faces" of the mesh), such that, for all $\sigma\in\mathcal{E}$, $\sigma$ is a non empty closed subset of a hyperplane of $\mathbb{R}^d$, which has a $(d-1)$-dimensional measure $\mathrm{m}_\sigma>0$. We assume that, for all $K\in\mathcal{T}$, there exists a subset $\mathcal{E}_K$ of $\mathcal{E}$ such that $\partial K=\bigcup_{\sigma\in\mathcal{E}_K}\sigma$. We then denote by $\mathcal{T}_\sigma$ the set $\{K\in\mathcal{T}\mid\sigma\in\mathcal{E}_K\}$. It is assumed that, for all $\sigma\in\mathcal{E}$, either $\mathcal{T}_\sigma$ has exactly one element and then $\sigma\subset\partial\Omega$ (boundary face) or $\mathcal{T}_\sigma$ has exactly two elements (interior face). For all $\sigma\in\mathcal{E}$, we denote by $x_\sigma$ the center of gravity of $\sigma$.
- $\mathcal{P}$ is a family of points of $\Omega$ indexed by $\mathcal{T}$ ("the centers of cells"), denoted by $\mathcal{P}=(x_K)_{K\in\mathcal{T}}$, such that $x_K\in K$ and $K$ is star-shaped with respect to $x_K$.
- $\mathcal{V}$ is a family of points ("the vertices of the mesh"), such that for any $K\in\mathcal{T}$, for all subset $H_K$ of $\mathcal{E}_K$ with cardinal$(H_K)\geqslant d$, then $\bigcap_{\sigma\in H_K}\sigma=\emptyset$ or $\bigcap_{\sigma\in H_K}\sigma=s$ where $s\in\mathcal{V}$. For all $s\in\mathcal{V}$, we denote by $\mathcal{E}_s$ the set $\{\sigma\in\mathcal{E}\mid s\in\sigma\}$ and by $\mathcal{T}_s$ the set $\{K\in\mathcal{T}\mid s\in\overline{K}\}$. For all $K\in\mathcal{T}$, the set $\mathcal{V}_K$ stands for $\{s\in\mathcal{V}\mid s\in\overline{K}\}$, and for all $\sigma\in\mathcal{E}$ the set $\{s\in\mathcal{V}\mid s\in\sigma\}$ is denoted by $\mathcal{V}_\sigma$.

Fig. 1 gives an example of a 2D admissible mesh and illustrate some of the above and following notations. The size of the discretization is defined by: $h_\mathcal{D}=\sup\{\mathrm{diam}(K),K\in\mathcal{T}\}$. For all $K\in\mathcal{T}$ and $\sigma\in\mathcal{E}_K$, we denote by $\mathbf{n}_{K,\sigma}$ the unit vector normal to $\sigma$ outward to $K$, and by $d_{K,\sigma}$ the Euclidean distance between $x_K$ and $\sigma$. The set of interior (resp. boundary) faces is denoted by $\mathcal{E}_{\mathrm{int}}$ (resp. $\mathcal{E}_{\mathrm{ext}}$), defined by $\mathcal{E}_{\mathrm{int}}=\{\sigma\in\mathcal{E}\mid\sigma\not\subset\partial\Omega\}$ (resp. $\mathcal{E}_{\mathrm{ext}}=\{\sigma\in\mathcal{E}\mid\sigma\subset\partial\Omega\}$). For any $K\in\mathcal{T}$ and $s\in\mathcal{V}_K$, $q_K^s$ stands for the cardinal of $\mathcal{E}_K\cap\mathcal{E}_s$.

**Shape regularity of the mesh.** It will be measured by the parameters regul($\mathcal{D}$) and ratio($\mathcal{D}$) defined by $\min_{\sigma\in\mathcal{E}_K,K\in\mathcal{T}}\{\frac{d_{K,\sigma}}{\mathrm{diam}(K)}\}$ and $\min_{\sigma\in\mathcal{E}_{\mathrm{int}},\mathcal{T}_\sigma=\{K,L\}}\{\frac{\min(d_{K,\sigma},d_{L,\sigma})}{\max(d_{K,\sigma},d_{L,\sigma})}\}$.

**Parameters of the MPFA O finite volume scheme.** In addition to the choice of the cell centers satisfying the above assumptions, the construction of the MPFA O scheme involves two families of parameters defined on the set $\{(\sigma,s)\mid s\in\mathcal{V}_\sigma,\sigma\in\mathcal{E}\}$. The first family of non negative reals $(\mathrm{m}_\sigma^s)_{s\in\mathcal{V}_\sigma,\sigma\in\mathcal{E}}$ defines the distribution of the "area" $\mathrm{m}_\sigma$ of each face $\sigma$ to the face vertices $s\in\mathcal{V}_\sigma$ such that $\mathrm{m}_\sigma=\sum_{s\in\mathcal{V}_\sigma}\mathrm{m}_\sigma^s$.

It results that the volume of each cell $K\in\mathcal{T}$ is also distributed to the vertices of the cell according to the subvolumes $\mathrm{m}_K^s$, $s\in\mathcal{V}_K$ defined by $\mathrm{m}_K^s=\frac{1}{d}\sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_s}\mathrm{m}_\sigma^s d_{K,\sigma}$, and which satisfy $\mathrm{m}_K=\sum_{s\in\mathcal{V}_K}\mathrm{m}_K^s$ for all $K\in\mathcal{T}$. The second family is the set of the so called continuity points $(x_\sigma^s)_{\sigma\in\mathcal{E}_s,s\in\mathcal{V}}$ such that $x_\sigma^s\in\sigma$. On each continuity point $x_\sigma^s$, the intermediate unknown $u_\sigma^s$ is defined which will be used together with the cell unknowns $u_K$, $K\in\mathcal{T}$ for the construction of the discrete gradients defined in Section 3.

**Discrete function spaces.** Let $\mathcal{H}_\mathcal{D}$ be the subspace of $\{(u_K)_{K\in\mathcal{T}}, (u_\sigma^s)_{\sigma\in\mathcal{E}_s, s\in\mathcal{V}}, u_K, u_\sigma^s \in \mathbb{R}\}$ such that $u_\sigma^s = 0$ for all $s\in\mathcal{V}_\sigma$, $\sigma\in\mathcal{E}_{\text{ext}}$. The space $\mathcal{H}_\mathcal{D}$ is equipped with the following Euclidean structure defined by the inner product: for $(v,w)\in(\mathcal{H}_\mathcal{D})^2$,

$$[v,w]_\mathcal{D} = \sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K}\sum_{s\in\mathcal{V}_\sigma}\frac{\mathrm{m}_\sigma^s}{d_{K,\sigma}}\big(v_\sigma^s - v_K\big)\big(w_\sigma^s - w_K\big) \tag{2}$$

and the associated norm: $\|u\|_\mathcal{D} = ([u,u]_\mathcal{D})^{1/2}$. Let $H_\mathcal{T}(\Omega)\subset L^2(\Omega)$ be the space of piecewise constant functions on each cell of the mesh $\mathcal{T}$, equipped with the following norm: $\|u\|_\mathcal{T} = \inf\{\|v\|_\mathcal{D}, v\in\mathcal{H}_\mathcal{D}, P_\mathcal{T}v = u\}$ where for all $v\in\mathcal{H}_\mathcal{D}$, $P_\mathcal{T}v\in H_\mathcal{T}(\Omega)$ denotes the vector of $H_\mathcal{T}(\Omega)$ defined by $(v_K)_{K\in\mathcal{T}}$.

## 3. The MPFA scheme and its discrete variational formulation

Let us define the discrete gradients $(\widetilde{\nabla}_\mathcal{D}v)_K^s = \frac{1}{\mathrm{m}_K^s}\sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_s}\mathrm{m}_\sigma^s(v_\sigma^s - v_K)\mathbf{n}_{K,\sigma}$, and $(\overline{\nabla}_\mathcal{D}v)_K^s = (B_K^s)^{-1}(\widetilde{\nabla}_\mathcal{D}v)_K^s$ for each $s\in\mathcal{V}_K$, and $K\in\mathcal{T}$, where the square matrix $B_K^s$ is given by

$$B_K^s = \frac{1}{\mathrm{m}_K^s}\sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_s}\mathrm{m}_\sigma^s\mathbf{n}_{K,\sigma}\big(x_\sigma^s - x_K\big)^t.$$

Note that the non singularity of the matrix $B_K^s$ for all $s\in\mathcal{V}_K$, $K\in\mathcal{T}$ will derive from the stronger coercivity condition stated in Section 4. Let $a_\mathcal{D}$ be the bilinear form defined on $\mathcal{H}_\mathcal{D}\times\mathcal{H}_\mathcal{D}$ by

$$a_\mathcal{D}(u,v) = \sum_{K\in\mathcal{T}}\sum_{s\in\mathcal{V}_K}\left(\mathrm{m}_K^s(\overline{\nabla}_\mathcal{D}u)_K^s\cdot\Lambda_K(\widetilde{\nabla}_\mathcal{D}v)_K^s + \alpha_K^s\sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_s}\frac{\mathrm{m}_\sigma^s}{d_{K,\sigma}}R_{K,\sigma}^s(u)R_{K,\sigma}^s(v)\right),$$

for all $u,v\in\mathcal{H}_\mathcal{D}$, where $\Lambda_K = \frac{1}{\mathrm{m}_K}\int_K\Lambda(x)\,\mathrm{d}x$, and the residual function $R_{K,\sigma}$ is defined by $R_{K,\sigma}^s(u) = u_\sigma^s - u_K - (\overline{\nabla}_\mathcal{D}u)_K^s\cdot(x_\sigma^s - x_K)$. Our scheme is defined by the following discrete hybrid variational formulation: find $u\in\mathcal{H}_\mathcal{D}$ such that $a_\mathcal{D}(u,v) = \int_\Omega f(P_\mathcal{T}v)$ for all $v\in\mathcal{H}_\mathcal{D}$.

Checking that $a_\mathcal{D}(u,v) = \sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K}\sum_{s\in\mathcal{V}_\sigma}F_{K,\sigma}^s(u)(v_K - v_\sigma^s)$, for all $u,v\in\mathcal{H}_\mathcal{D}$ with the following definition of the subfluxes

$$F_{K,\sigma}^s(u) = -\mathrm{m}_\sigma^s\Lambda_K(\overline{\nabla}_\mathcal{D}u)_K^s\cdot\mathbf{n}_{K,\sigma} - \alpha_K^s\mathrm{m}_\sigma^s\left(\frac{R_{K,\sigma}^s(u)}{d_{K,\sigma}} - \frac{(B_K^s)^{-1}\mathbf{n}_{K,\sigma}}{\mathrm{m}_K^s}\cdot\sum_{\sigma'\in\mathcal{E}_K\cap\mathcal{E}_s}\frac{\mathrm{m}_{\sigma'}^s}{d_{K,\sigma'}}R_{K,\sigma'}^s(u)\big(x_{\sigma'}^s - x_K\big)\right),$$

for all $s\in\mathcal{V}_\sigma$, $\sigma\in\mathcal{E}_K$, $K\in\mathcal{T}$, it is easily shown that the hybrid variational formulation is equivalent to the following hybrid finite volume scheme: find $u\in\mathcal{H}_\mathcal{D}$ such that

$$\begin{cases} -\displaystyle\sum_{\sigma\in\mathcal{E}_K}F_{K,\sigma}(u) = \int_K f(x)\,\mathrm{d}x & \text{for all } K\in\mathcal{T},\\ F_{K,\sigma}(u) = \displaystyle\sum_{s\in\mathcal{V}_\sigma}F_{K,\sigma}^s(u) & \text{for all } \sigma\in\mathcal{E}_K, K\in\mathcal{T},\\ F_{K,\sigma}^s(u) + F_{L,\sigma}^s(u) = 0 & \text{for all } s\in\mathcal{V}_\sigma, \mathcal{T}_\sigma = \{K,L\}, \sigma\in\mathcal{E}_{\text{int}}. \end{cases} \tag{3}$$

Around each vertex $s\in\mathcal{V}$, the face unknowns $(u_\sigma^s)_{\sigma\in\mathcal{E}_s}$ can be eliminated in terms of the $(u_K)_{K\in\mathcal{T}_s}$ solving the linear system

$$\begin{cases} F_{K,\sigma}^s(u) + F_{L,\sigma}^s(u) = 0 & \text{for all } \sigma\in\mathcal{E}_s\cap\mathcal{E}_{\text{int}} \text{ with } \mathcal{T}_\sigma = \{K,L\},\\ u_\sigma^s = 0 & \text{for all } \sigma\in\mathcal{E}_s\cap\mathcal{E}_{\text{ext}}. \end{cases} \tag{4}$$

The well-posedness of this system derives from the coercivity condition stated below in Section 4. It results that the hybrid finite volume scheme reduces to a cell centered finite volume scheme for which the flux at each face $\sigma\in\mathcal{E}$ is a linear combination of the cell unknowns $u_M$ with $M\in\bigcup_{s\in\mathcal{V}_\sigma}\mathcal{T}_s$.

For all $s\in\mathcal{V}_K$, $K\in\mathcal{T}$, let us assume that $q_K^s = d$, and that both sets $(x_\sigma^s - x_K)_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_s}$ and $(\mathbf{n}_{K,\sigma})_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_s}$ span $\mathbb{R}^d$. Then, it can be shown that our finite volume scheme (3) is equivalent to the usual MPFA O scheme, since each discrete gradient $(\overline{\nabla}_\mathcal{D}u)_K^s$ matches with the gradient of the linear function uniquely defined by the $d+1$ points $(x_\sigma^s, u_\sigma^s)_{\sigma\in\mathcal{E}_s}$, $(x_K, u_K)$, and each residual $R_{K,\sigma}^s(u)$ vanishes for all $u\in\mathcal{H}_\mathcal{D}$. In addition the above formulation of the scheme provides a generalization of the scheme described in [1,6] for more general polyhedral meshes.

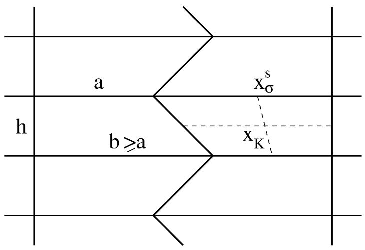


Fig. 2. Example of a trapezoidal mesh.

## 4. Coercivity and convergence of the MPFA O scheme

In order to obtain existence, uniqueness of the solution and stability estimates, a coercivity property is needed in the sense that there exists a real $\beta > 0$ such that, for all $u \in \mathcal{H}_\mathcal{D}$, $a_\mathcal{D}(u,u) \geqslant \beta \|u\|_\mathcal{D}^2$.

This is achieved imposing the following sufficient condition: there exists a real $\theta > 0$ such that

$$\mathrm{coer}(\mathcal{D}, \Lambda) \geqslant \theta, \tag{5}$$

where $\mathrm{coer}(\mathcal{D}, \Lambda)$ is defined by

$$\mathrm{coer}(\mathcal{D}, \Lambda) = \min_{K \in \mathcal{T},\ s \in \mathcal{V}_K} \lambda_{\min}\big(\Lambda_K B_K^s + (\Lambda_K B_K^s)^t\big). \tag{6}$$

This condition can be easily computed for any given finite volume discretization $\mathcal{D}$ and diffusion tensor $\Lambda$. Assuming that this condition holds uniformly, the following theorem is proved in [4]:

**Theorem 4.1** *(Convergence of the scheme). Let $(\mathcal{D}^{(n)})_{n\in\mathbb{N}}$ be a family of finite volume discretization, let $u_{\mathcal{D}^{(n)}} \in \mathcal{H}_{\mathcal{D}^{(n)}}$ be such that* (3) *holds, let* $\mathrm{regul}(\mathcal{D}^{(n)}) \geqslant \beta$ *for some $\beta > 0$,* $\mathrm{ratio}(\mathcal{D}^{(n)}) \geqslant \zeta$ *for some $\zeta > 0$ and* $\mathrm{coer}(\mathcal{D}^{(n)}, \Lambda) \geqslant \theta$ *for some $\theta > 0$. Then, $P_\mathcal{T} u_{\mathcal{D}^{(n)}}$ converges to $\bar{u}$ in $L^q(\Omega)$, for all $q \in [1, +\infty)$ if $d = 2$ and all $q \in [1, 2d/(d-2))$ if $d > 2$, as $h_{\mathcal{D}^{(n)}} \to 0$. Moreover, $\widehat{\nabla}_{\mathcal{D}^{(n)}} u_{\mathcal{D}^{(n)}} \in H_\mathcal{T}(\Omega)^d$ defined by $\mathrm{m}_K(\widehat{\nabla}_{\mathcal{D}^{(n)}} u_{\mathcal{D}^{(n)}})_K = \sum_{s\in\mathcal{V}_K} \mathrm{m}_K^s (\overline{\nabla}_{\mathcal{D}^{(n)}} u)_K^s$ for all $K \in \mathcal{T}$, converges to $\nabla\bar{u}$ in $L^2(\Omega)^d$.*

## 5. Numerical examples

Let us first discuss the coercivity condition (5) on a few examples. For all $\sigma \in \mathcal{E}_K \cap \mathcal{E}_s$ let us choose $m_\sigma^s = \frac{m_\sigma}{\mathrm{cardinal}(\mathcal{V}_\sigma)}$, $x_K$ the isobarycenter of the vertices of the cell $K$ and let $x_\sigma^s$ be the center of gravity of the face $\sigma$. Then, for parallelogram and parallelepiped cells, the matrix $B_K^s$ is equal to $I$. In such a case, the MPFA O scheme is symmetric and our sufficient condition of coercivity (5) is always satisfied. The same result holds for triangles with $x_\sigma^s$ the barycenter with weights 2/3 at point $s$ and 1/3 at the second end point of the edge $\sigma$. It holds again for tetrahedrons with $x_\sigma^s$ the barycenter with weights 1/2 at point $s$ and 1/4 at the two remaining vertices of the face $\sigma$.

Let us now consider the case $d = 2$ with $\Lambda = I$, and let $\sigma_1$ and $\sigma_2$ be the two edges shared by a given vertex $s$ of a given cell $K$. For $\sigma = \sigma_1, \sigma_2$, we assume that the continuity point $x_\sigma^s$ is the center of gravity $x_\sigma$ of the edge $\sigma$ and that $m_\sigma^s = |x_\sigma - s|$. Then, the condition $\lambda_{\min}(B_K^s + (B_K^s)^t) \geqslant \theta$ is equivalent to $|x_{\sigma_1} - x_{\sigma_2}|\, |\overrightarrow{sx_{\sigma_1}} - \overrightarrow{x_{\sigma_2}x_K}| \leqslant 2(1 - \frac{\theta}{2}) m_K^s$.

For example, the trapezoidal mesh exhibited Fig. 2 satisfies the coercivity condition (5) if and only if $\frac{b-a}{h} \leqslant (1 - \frac{\theta}{2}) \frac{3a+b}{(b^2+h^2)^{1/2}}$ which exhibits the lack of robustness of the MPFA O scheme for distorted quadrangular meshes. Next, in the following test case, the MPFA O scheme is compared with a symmetric coercive scheme presented in [3] on a family of 2D distorted quadrangular meshes of the domain $\Omega = (0,1)^2$, and for an homogeneous anisotropic diffusion tensor $K = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & 1 \end{smallmatrix}\right)$, with midle anisotropy $\lambda = 10$ and high anisotropy $\lambda = 1000$. The meshes are obtained by random distorsion of the uniform Cartesian meshes of size $n_x \times n_x$ with $n_x = 4, 8, 16, 32, 64, 128$ (see Fig. 3 for $n_x = 16$). The right-hand side $f$ of the diffusion equation (1) is defined in order to obtain the exact solution $u(x,y) = \sin(\pi x)\sin(\pi y)$.

Fig. 3 exhibits that the MPFA O scheme is more accurate then the symmetric unconditionally coercive scheme [3] on mildly anisotropic test cases but it lacks robustness due to the breakdown of the coercivity for problems combining

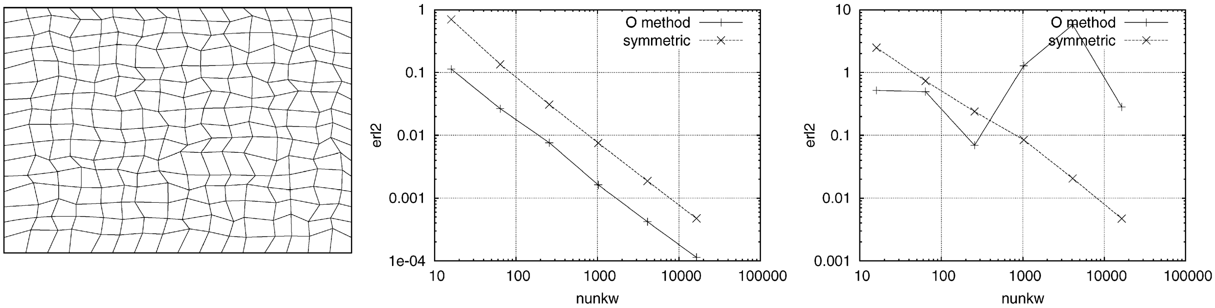


Fig. 3. Mesh of size $n_x = 16$, and convergence of the $L^2$ error (erl2) for the MPFA O scheme and the Symmetric scheme for $\lambda = 10$ (left), and $\lambda = 1000$ (right) (nunkw denotes the number of cells $n_x^2$).

both distorted meshes and high anisotropy. However, this additional robustness of the symmetric coercive scheme [3] is obtained at the price of much larger flux and scheme stencils, and the MPFA O scheme exhibits a good compromise between robustness and CPU time efficiency.

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