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Numerical Analysis

Discontinuous Galerkin approximation with discrete variational principle for the nonlinear Laplacian

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Abstract

A discontinuous Galerkin method is analyzed to approximate the nonlinear Laplacian model problem. The salient feature of the proposed scheme is that it is endowed with a discrete variational principle. The convergence of the discrete approximations to the exact solution is proven. ***To cite this article: E. Burman, A. Ern, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Approximation par la méthode de Galerkine discontinue avec un principe variationnel discret pour un Laplacien non-linéaire. On analyse une méthode de Galerkine discontinue afin d'approcher le problème modèle du Laplacien non-linéaire. La propriété essentielle du schéma proposé est que celui-ci jouit d'un principe variationnel discret. On prouve la convergence des approximations discrètes vers la solution exacte. ***Pour citer cet article : E. Burman, A. Ern, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

1. Introduction

Let Ω be a open bounded connected subset of $\mathbb{R}^d$ whose boundary is a finite union of parts of hyperplanes. Let $1 < p < +\infty$. Consider the nonlinear variational problem consisting of finding the minimizer in $W_0^{1,p}(\Omega)$ of the functional

$$J : W_0^{1,p}(\Omega) \ni v \longmapsto \frac{1}{p}\int_\Omega |\nabla v|^p - \int_\Omega f v \in \mathbb{R}, \tag{1}$$

where $f \in L^q(\Omega)$ with $q = \frac{p}{p-1}$ and where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^d$. It is well-known (see, e.g., [7] and [4, p. 312]) that the functional J admits a unique minimizer $u \in W_0^{1,p}(\Omega)$. This minimizer solves the associated

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Euler–Lagrange optimality conditions stating that

$$\forall v \in W_0^{1,p}(\Omega), \quad \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v = \int_\Omega f v. \tag{2}$$

This so-called nonlinear Laplacian is a prototype for problems encountered, for instance, with filtration models or quasi-Newtonian flows.

The finite element approximation of the nonlinear Laplacian model problem has been considered previously by various authors; see, e.g., [1,3] for some of the earlier works using conforming finite elements and [8] for a more recent work using nonconforming finite elements. Discontinuous Galerkin (dG) methods have received extensive interest over the last decade, in particular because of their flexibility in the construction of computational meshes and the use of variable polynomial degrees. However, to the best of our knowledge, a dG method has not yet been devised to approximate (2). The purpose of this Note is to fill this gap. A salient feature of the proposed method is its discrete variational principle, meaning that the discrete solution minimizes over the discrete dG space a consistent modification of the nonlinear energy functional (1). To achieve this property, we use a discrete gradient reconstruction, which from a mathematical viewpoint, satisfies an important compactness property recently established in [5]. Such discrete gradients arise naturally in the formulation of dG approximations and have been used recently also in [6] for nonlinear elasticity and in [2] for nonlinear diffusion.

This Note is organized as follows. In Section 2 we formulate the dG approximation. In Section 3 we prove the convergence of the method. The main result is Theorem 3.1.

2. Formulation of the dG approximation

The family $\{\mathcal{T}_h\}_{h\in\mathcal{H}}$, where $\mathcal{H}$ is a countable set, is said to be an *admissible* mesh family if the following assumptions are satisfied:

(i) for all $h \in \mathcal{H}$, $\mathcal{T}_h$ is a finite family of nonempty convex (possibly nonconvex) open disjoint sets T forming a partition of Ω and whose boundaries are a finite union of parts of hyperplanes;
(ii) there is a parameter N_∂, independent of h, such that each $T \in \mathcal{T}_h$ has at most N_∂ faces, where a set $F \subset \partial T$ is said to be a face of T if F is part of a hyperplane, and if either $F = \partial T \cap \partial\Omega$ or there is $T' \in \mathcal{T}_h$ such that $F = \partial T \cap \partial T'$;
(iii) there is a parameter ϱ_1 independent of h such that for all $T \in \mathcal{T}_h$, $\sum_{F\subset\partial T} h_F|F| \leqslant \varrho_1|T|$, where h_F denotes the diameter of the face F, $|F|$ its $(d-1)$-dimensional measure and $|T|$ the d-dimensional measure of T;
(iv) for all $h \in \mathcal{H}$, each $T \in \mathcal{T}_h$ is affine-equivalent to an element of a finite collection of reference elements, and there is parameter ϱ_2, independent of h, bounding the ratio of the diameter h_T of any $T \in \mathcal{T}_h$ to the diameter of the largest ball inscribed in T.

For each $h \in \mathcal{H}$, we define $\mathrm{size}(\mathcal{T}_h) \stackrel{\text{def}}{=} \max_{T\in\mathcal{T}_h} h_T$. The mesh parameters N_∂, ϱ_1, ϱ_2 and the reference elements will be collectively denoted by the symbol $\mathcal{P}$. The mesh faces are collected in the set $\mathcal{F}_h$. It will be convenient to partition the set $\mathcal{F}_h$ into $\mathcal{F}_h^i \cup \mathcal{F}_h^b$ where $\mathcal{F}_h^b$ collects the faces located on the boundary of Ω and $\mathcal{F}_h^i$ collects the remaining ones. For $F \in \mathcal{F}_h^i$, there are T_1 and T_2 in $\mathcal{T}_h$ such that $F = \partial T_1 \cap \partial T_2$, and we define ν_F as the unit normal vector to F pointing from T_1 to T_2. For any function φ such that a (possibly two-valued) trace is defined on F, let $[\![\varphi]\!] \stackrel{\text{def}}{=} \varphi_{|T_1} - \varphi_{|T_2}$ and $\{\{\varphi\}\} \stackrel{\text{def}}{=} \frac{1}{2}(\varphi_{|T_1} + \varphi_{|T_2})$. For $F \in \mathcal{F}_h^b$, ν_F is defined as the unit outward normal to Ω, while the jump and average are conventionally defined as $[\![\varphi]\!] \stackrel{\text{def}}{=} \varphi$ and $\{\{\varphi\}\} \stackrel{\text{def}}{=} \varphi$.

Consider the finite dimensional space $V_h \stackrel{\text{def}}{=} \{v_h \in L^2(\Omega);\ \forall T \in \mathcal{T}_h,\ v_{h|T} \in \mathbb{P}_1(T)\}$ spanned by piecewise affine polynomials. This space is equipped with the norm

$$\|v_h\|_{\mathrm{DG},p}^p \stackrel{\text{def}}{=} \sum_{T\in\mathcal{T}_h} \int_T |\nabla v_h|^p + \sum_{F\in\mathcal{F}_h} \frac{1}{h_F^{p-1}} \int_F \big|[\![v_h]\!]\big|^p. \tag{3}$$

Discrete Sobolev embeddings that are the exact counterpart of those holding in $W_0^{1,p}(\Omega)$ are proven in [5] on the spaces V_h using the norm defined by (3) (actually, an equivalent variant using $|\nabla v_h|_{\ell^p}^p \stackrel{\text{def}}{=} \sum_{i=1}^d |\partial_i v_h|^p$ instead of

$|\nabla v_h|^p$). Here, we shall use the fact that there is σ depending on p and $\mathcal{P}$ s.t.

$$\forall v_h \in V_h, \quad \|v_h\|_{L^p(\Omega)} \leqslant \sigma \|v_h\|_{\mathrm{DG},p}. \tag{4}$$

For all $F \in \mathcal{F}_h$, let $r_F : L^2(F) \to [V_h^0]^d$ be the lifting operator such that for all $\phi \in L^2(F)$, $r_F(\phi)$ is defined s.t. $\forall \tau_h \in [V_h^0]^d$, $\int_\Omega r_F(\phi) \cdot \tau_h = \int_F \{\{\tau_h\}\} \cdot \nu_F \phi$, where V_h^0 denotes the space spanned by piecewise constant functions on $\mathcal{T}_h$. Clearly, the support of $r_F(\phi)$ consists of the one or two mesh elements of which F is a face. Then, define the discrete gradient operator $G_h : V_h \to [V_h]^d$ s.t.

$$G_h(v_h) \stackrel{\text{def}}{=} \nabla_h v_h - \sum_{F \in \mathcal{F}_h} r_F\big([\![v_h]\!]\big), \tag{5}$$

where $\nabla_h v_h$ denotes the usual broken gradient of v_h. The following two results, proven in [5], state the key stability and compactness properties of the discrete gradient:

Lemma 2.1 *(Stability). For all $v_h \in V_h$,*

$$\big\|\nabla_h v_h - G_h(v_h)\big\|_{L^p(\Omega)^d}^p \leqslant N_\partial^{p-1} \sum_{F \in \mathcal{F}_h} \big\|r_F\big([\![v_h]\!]\big)\big\|_{L^p(\Omega)^d}^p, \tag{6}$$

and there is c (depending on k and $\mathcal{P}$) such that for all $F \in \mathcal{F}_h$, $\|r_F([\![v_h]\!])\|_{L^p(\Omega)^d}^p \leqslant c h_F^{1-p} \int_F |[\![v_h]\!]|^p$.

Lemma 2.2 *(Compactness). Let $\{v_h\}_{h \in \mathcal{H}}$ be a sequence in V_h and assume that this sequence is bounded in the $\|\cdot\|_{\mathrm{DG},p}$-norm. Assume that $\mathrm{size}(\mathcal{T}_h) \to 0$. Then, there exists $v \in W_0^{1,p}(\Omega)$ such that, up to a subsequence, $v_h \to v$ in $L^p(\Omega)$ and $G_h(v_h) \rightharpoonup \nabla v$ weakly in $L^p(\Omega)^d$.*

Define the discrete functional

$$J_h : V_h \ni v_h \longmapsto \frac{1}{p} \int_\Omega \big|G_h(v_h)\big|^p + \frac{1}{p} \sum_{F \in \mathcal{F}_h} \eta \frac{1}{h_F^{p-1}} \int_F \big|[\![v_h]\!]\big|^p - \int_\Omega f v_h \in \mathbb{R}, \tag{7}$$

where $\eta > 0$ is a (user-dependent) parameter. The dG approximation u_h is then defined as the minimizer over the discrete space V_h of the discrete functional J_h. It is easily verified that this minimizer exists and is unique. Equivalently, it satisfies the following Euler–Lagrange optimality conditions $a_h(u_h, v_h) = \int_\Omega f v_h$, $\forall v_h \in V_h$, with

$$a_h(u_h, v_h) = \int_\Omega \big|G_h(u_h)\big|^{p-2} G_h(u_h) \cdot G_h(v_h) + \sum_{F \in \mathcal{F}_h} \eta \frac{1}{h_F^{p-1}} \int_F \big|[\![u_h]\!]\big|^{p-2} [\![u_h]\!][\![v_h]\!]. \tag{8}$$

The discrete Euler–Lagrange equations amount to solving a finite set of coupled nonlinear equations. It is readily verified using Lemma 2.1 that the semi-linear form a_h verifies the following coercivity property: there is $\alpha > 0$, depending on p, $\mathcal{P}$ and η s.t.

$$\forall v_h \in V_h, \quad a_h(v_h, v_h) \geqslant \alpha \|v_h\|_{\mathrm{DG},p}^p. \tag{9}$$

3. Convergence analysis

Theorem 3.1 *(Convergence). Let $\{u_h\}_{h \in \mathcal{H}}$ be the sequence of approximate solutions generated by solving the discrete problems on the admissible meshes $\{\mathcal{T}_h\}_{h \in \mathcal{H}}$. Then, as $\mathrm{size}(\mathcal{T}_h) \to 0$,*

$$u_h \to u, \quad \text{in } L^p(\Omega), \tag{10}$$

$$G_h(u_h) \to \nabla u, \quad \text{in } L^p(\Omega)^d, \tag{11}$$

and $\sum_{F \in \mathcal{F}_h} \frac{1}{h_F^{p-1}} \int_F |[\![u_h]\!]|^p \to 0$, where $u \in W_0^{1,p}(\Omega)$ is the global minimizer of the functional J in $W_0^{1,p}(\Omega)$.

Proof. (i) A priori estimate. Owing to (9), the Sobolev embedding (4) and Hölder's inequality, it is readily inferred that

$$\|u_h\|_{\mathrm{DG},p} \leqslant \left(\alpha^{-1}\sigma\|f\|_{L^q(\Omega)}\right)^{1/(p-1)}.$$

Lemma 2.2 then implies that there exists $u \in W_0^{1,p}(\Omega)$ such that, up to a subsequence, $u_h \to u$ in $L^p(\Omega)$ and $G_h(u_h) \rightharpoonup \nabla u$ weakly in $L^p(\Omega)^d$ as $\mathrm{size}(\mathcal{T}_h) \to 0$.

(ii) Identification of the limit. Owing to weak convergence and convexity, $\liminf \int_\Omega |G_h(u_h)|^p \geqslant \int_\Omega |\nabla u|^p$, and since the jump term in J_h is nonnegative, this yields $\liminf J_h(u_h) \geqslant J(u)$. Let now $\varphi \in C_c^\infty(\Omega)$ and let $\pi_h\varphi$ denote the L^2-orthogonal projection of φ onto V_h. Owing to standard approximation properties, $\|\varphi - \pi_h\varphi\|_{\mathrm{DG},p} \to 0$ as $\mathrm{size}(\mathcal{T}_h) \to 0$. Hence, by continuity, $\liminf J_h(\pi_h\varphi) = J(\varphi)$. Since $J_h(\pi_h\varphi) \geqslant J_h(u_h)$ by construction, this implies $J(u) \leqslant J(\varphi)$, and by density of $C_c^\infty(\Omega)$ in $W_0^{1,p}(\Omega)$, this shows that u is a minimizer of the exact problem. Since this minimizer is unique, the whole sequence $\{u_h\}_{h\in\mathcal{H}}$ converges to u (strongly in $L^p(\Omega)$) and the whole sequence $\{G_h(u_h)\}_{h\in\mathcal{H}}$ converges to ∇u (weakly in $L^p(\Omega)^d$).

(iii) Strong convergence of the gradient. Observing that

$$\limsup \left\|G_h(u_h)\right\|^p_{L^p(\Omega)^d} \leqslant \limsup a_h(u_h,u_h) = \limsup \int_\Omega f u_h = \int_\Omega f u = \|\nabla u\|^p_{L^p(\Omega)^d},$$

this classically implies the strong convergence of the gradient. Similarly, it is inferred that $a_h(u_h,u_h) \to \|\nabla u\|^p_{L^p(\Omega)^d}$, so that

$$\sum_{F\in\mathcal{F}_h} \eta \frac{1}{h_F^{p-1}} \int_F \left|[\![u_h]\!]\right|^p = a_h(u_h,u_h) - \left\|G_h(u_h)\right\|^p_{L^p(\Omega)^d},$$

converges to zero. The proof is complete. □

Preliminary numerical results in one space dimension indicate a convergence rate of at least $h^{3/4}$ for $p \in \{3,4,5\}$ and smooth exact solution.

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