Abstract

We extend the method of Zariski to determine the braid monodromy group of the discriminant of a versal unfolding of a hypersurface singularity from low-dimensional generic subunfoldings to highly non-generic ones. At the expense of an induction over adjacent singularities, it is thus possible to neglect genericity issues and perturb by very simple polynomials only. To cite this article: M. Lönne, C. R. Acad. Sci. Paris, Ser. I 346 (2008).

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1. Introduction

Due to a wealth of remarkable properties the versal unfolding of a hypersurface singularity $f$ has been a challenge for a long time. It governs the Milnor fibre, the intersection lattice and the monodromy of $f$. These data are also obtained from a Morsification or inductively from singularities adjacent to $f$, cf. [1,3].

We are interested in the braid monodromy group of discriminant complements of versal unfoldings. This invariant of $f$ is given also by discriminant complements of suitable subunfoldings, which we call Zariskification, since their existence is due to Zariski’s result on generic hypersurface sections.

In this Note we define the notion of versal braid monodromy group for any subunfolding, which takes into account the braid monodromies of versal unfoldings of adjacent singularities, and we show that it coincides with the braid monodromy group of $f$ under a weak transversality condition, cf. [2].

We thus gain much flexibility in the choice of subunfoldings, which has been exploited with great success in the computation [6,5] of the braid monodromy and the fundamental group of discriminant complements for Brieskorn–Pham polynomials. (In this Note we generally employ the notions and notations of [1].)
2. Distinguished classes of unfoldings

It is well known that a Morsification of an isolated singularity $f \in \mathcal{O}_n$ can be understood as an unfolding

$$ f_{t,u} : \mathbb{C}^n \times \mathbb{C}^2 \to \mathbb{C}, \quad x, t, u \mapsto f_{t,u}(x) = f_t(x) - u, \quad f_0(x) = f(x), $$

of hypersurfaces, induced by a map $\hat{\phi} : \mathbb{C}^2 \to \mathbb{C}^n \cong \mathcal{O}_n/J_f$ to the base of a miniversal unfolding with image of $\hat{\phi}$ restricted to constant $t_0 \neq 0$ transversal to the discriminant $D$, i.e. transversal to $D_{\text{reg}}$ and disjoint from $D_{\text{sing}}$.

In analogy we want to define a Zariskification to be an unfolding of functions

$$ f_{s,t} : \mathbb{C}^n \times \mathbb{C}^2, \quad x, s, t \mapsto f_{s,t}(x), \quad f_{0,0}(x) = f(x), \quad f_{s,t}(0) = 0, \quad \frac{\partial f}{\partial t}(0) \neq 0, $$

which is induced by a map $\phi : \mathbb{C}^2, 0 \to \mathbb{C}^{n-1} \cong \mathbb{C}_n/J_f$ to the base of a truncated miniversal unfolding such that the image of $\phi$ restricted to constant $s_0 \neq 0$ is transversal to the function bifurcation set $B$.

There is also an analogue for the fact that Morsifications determine the monodromy group of $f$:

**Lemma 2.1.** The braid monodromy group of a Zariskification is equal to the braid monodromy group.

**Proof.** It suffices to recall the analogous argument for Morsifications. The essential input is, that a curve $\im \phi|_{s=s_0}$, $s_0 \neq 0$ is transversal to $B$ and therefore the induced map on fundamental groups surjects. $\square$

Next we define versal braid monodromy groups for unfoldings $F : \mathbb{C}^n \times \mathbb{C}^k \to \mathbb{C}$, which are $B$-transversal in the sense that the inducing map $\phi_F$ does not map all tangents at 0 to the Zariski tangent space of $B$. For sake of clarity we restrict ourselves to tame unfoldings, where outside codimension two, each function is tame, i.e. critical values may only coincide for non-degenerate critical points.

Given a family of functions $f_\lambda$ with $f_0$ tame and $f_\lambda$ Morse for $\lambda \neq 0$ the associated family $p_\lambda$ of discriminant polynomials consists of monic, univariate polynomials which only have simple zeroes for $\lambda \neq 0$:

$$ p_\lambda(u) = 0 \iff \exists x : \nabla f_\lambda(x) = 0, \quad f_\lambda(x) = u. $$

Let $v_j$ denote the roots of $p_0$. Then for $\varepsilon > 0$ and $0 < \delta \ll \varepsilon$ sufficiently small, the discriminant complement $Y = \mathbb{C} \times D_\delta \setminus p_\lambda^{-1}(0)$ is trivializable over the disc $D_\delta$ in the complement of $\bigcup_j B_\varepsilon(v_j)$, Fig. 1.

In any fibre $Y_\lambda$, $0 < |\lambda| < \delta$ we assign a group of mapping classes choosing generators – for each $v_j$ – supported on punctured discs $D_j = Y_\lambda \cap B_\varepsilon(v_j)$:

- In case that $v_j$ is a multiple root of $p_0$, which is the image of a single critical point $c_j$ of $f$, we assign the braid monodromy group for the germ of $f$ at $c_j$ consisting of mapping classes supported on $D_j$.
- In case $v_j$ is the image of non-degenerate critical points of $f$, we choose the group of mapping classes of $D_j$ which fix the punctures and thus correspond to pure braids.

Given a tame $B$-transversal unfolding we assign groups of mapping classes to each tame function using local slices to the bifurcation locus. The versal braid monodromy group is then defined to be generated by all such classes.

![Fig. 1. Polynomial family.](image-url)
transported along all possible paths to a reference fibre. It is determined as a subgroup of the braid group $B_n$ up to conjugacy.

We have a result simplifying the computation of versal braid monodromy groups:

**Proposition 2.2.** If a family of polynomials is induced from a $B$-transversal unfolding and induces a surjection of fundamental groups of bifurcation complements, the versal braid monodromy groups are isomorphic.

**Proof.** Choose compatible base points. Given a generator $\beta$ in the versal braid monodromy there exists by definition a disc slicing a component $B$ of the bifurcation locus and a path $\gamma$ in the bifurcation complement such that $\beta$ is obtained by transport along $\gamma$ from a mapping class $\phi$ associated to the family over the disc. By surjectivity there is another disc transversal to $B$ and a path $\gamma'$ which both lift to the induced family. Moreover there is an arc $\alpha$ arbitrarily close to $B_{\text{reg}}$. Stratified isotopy shows that the versal braid monodromies of the discs are identified under transport along $\alpha$. Thus there is a braid $\beta'$ obtained from $\phi$ by transport along $\alpha$ and $\gamma'$ which is in the versal braid monodromy of the induced family. Since $\beta$ and $\beta'$ are equal up to transport along the composite of $\alpha$ with both paths, also $\beta$ is in the monodromy of the induced family by the surjectivity property again. 

3. Comparison of braid monodromies

**Theorem 3.1.** The braid monodromy group of a function $f$ is equal to the versal braid monodromy group of any of its $B$-transversal unfoldings which is tame.

**Proof.** Given a $B$-transversal unfolding we may induce another one meeting the hypotheses of Proposition 2.2

$$f_{s,t}: \mathbb{C}^n \times \mathbb{C}^2 \to \mathbb{C}, \quad f_{0,0}(x) = f(x), \quad f_{s,t}(0) = 0, \quad \frac{\partial f}{\partial t}(0) \notin T_{\text{Zar}} B,$$

which thus has the same versal braid monodromy. By the condition on the unfolding $f_{0,t} + sg$ is a Zariskification for generic $g \in m_n$. Hence for all sufficiently small $\varepsilon \neq 0$ the two-parameter family $F = f_{\varepsilon,t} + sg$ has the same braid monodromy as any versal unfolding of $f$ while the one-parameter family $F|_{s=0} = f_{\varepsilon,t}$ has the same versal braid monodromy as the given $B$-transversal unfolding.

For each point $y_j$ in the bifurcation set on the line $s = 0$ let $U_j$ be a small ball in the base of $F$ centered at $y_j$. We fix a sufficiently small tubular neighbourhood $N_g$ of the line $s = 0$, such that the bifurcation set of $F|_{N_g}$ is in the union of the $U_j$ with singular locus in a subset of the $y_j$. The braid monodromy of $F|_{N_g}$ is thus equal to the braid monodromy of $f$. On the other hand it is generated by the braid monodromies of the $F|_{U_i}$ and parallel transport over the complement of the $U_i$. This should be compared to the fact that the versal braid monodromy of $F|_{s=0}$ is generated by the versal braid monodromies of $F|_{E_i}$ where $E_i$ denotes the intersection of $s = 0$ with $U_i$ and parallel transport over the complement of the $E_i$.

So it remains to prove that the versal braid monodromy of $F|_{E_i}$ is equal to the braid monodromy of $F|_{U_i}$ for each $i$, since the complement of the $U_i$ in $N_g$ retracts onto the complement of the $E_i$ on $s = 0$.

Let us thus consider a single ball $U$ and the discriminant family of $F|_{U_i}$. Its restriction to $E$ is a discriminant family with a single singular fibre and has a local description as in Section 2 which extends for $U$ sufficiently small. The complement $Y$ of the discriminant in $C \times U$ is trivializable over $U$ in the complement of balls $B_e(v_j)$ centered at the roots $v_j$ on the fibre over $y$.

The braid monodromy of $F|_U$ and the versal braid monodromy of $F|_{E}$ can thus be considered as a group of mapping classes which are supported on the intersection $\bigcup_j D_j$ of a local Milnor fibre with $\bigcup_j B_e(v_j)$.

According to the decomposition of the discriminant into connected components $D_j$ over $U$, the bifurcation locus decomposes, $B = \bigcup_j B_j$, where each $B_j$ is the branch locus of the finite map of $D_j$ onto $U$. Since the $B_e(v_j)$ are disjoint, the braid monodromy transformation along a simple geometric element based at the chosen Milnor fibre and associated to $B_j$ can be chosen with support in $D_j$.

Consider first a root $v_j$ which is the value of non-degenerate critical points of the function $f_y$. Its local discriminant $D_j$ in $B_e(v_j)$ has smooth branches in bijection to the preimages. Hence all mapping classes in the braid monodromy of $F|_{U}$ restrict to mapping classes of $D_j$ which fix the punctures pointwise.

On the other hand $E' := U \cap \{s = \eta\}$ is transversal to the bifurcation set, so the divisorial discriminant components in $B_e(v_j)$ meet pairwise, transversally, and over distinct points of the bifurcation set $B_j \cap E'$. This implies that the braid
monodromy of $F|_{E'}$ contains all pure mapping classes of $D_j$, i.e. the group of mapping classes which are supported on $D_j$ and fix the punctures pointwise. Hence this braid monodromy contains all mapping classes we assign to $v_j$ to get the versal braid monodromy group of $F|_{E}$.

Similarly we argue in case the root $v_j$ is the value of a unique critical point $c_j$. Then $B_\varepsilon(v_j)$ can be considered as a discriminant family induced from the base of a versal truncated unfolding of the function at $c_j$. It is in fact a Zariskification, see Fig. 2, since its bifurcation set $B_j$ is met by $E$ in a single point only and transversally by $E'$. Hence the braid monodromy of $F|_{U}$ contains the braid monodromy of the function at $c_j$ considered as mapping classes on $D_j$ extended by the identity to the Milnor fibre of $F|_{U}$, which is just what we assigned to $v_j$ to get the versal braid monodromy group.

Note that with a straightforward generalisation of versal braid monodromy to the non-tame case this claim holds in general. To demonstrate its power we show how our method is exploited in the case of a Brieskorn–Pham polynomial, cf. [5]. By the following proposition the inductive regress is through Brieskorn–Pham polynomials of decreasing codimension only which ends at polynomials of type $A_k$, and we may work with linear unfoldings only, considered in [4], where explicit formulae are known for the discriminant:

**Proposition 3.2.** The braid monodromy group of a Brieskorn Pham singularity given by $f(x) = \sum_i x_i^{l_i}$ is generated by the versal braid monodromy groups of the families

$$f_\lambda: \quad x \mapsto f(x) - \lambda x_1 - \sum_{i>1} \varepsilon_i x_i, \quad \text{and} \quad g_\alpha|_{|\alpha|\leq 1}: \quad x \mapsto f(x) - x_1 - \alpha \sum_{i>1} \varepsilon_i x_i,$$

where $0 < \varepsilon_2, \ldots, \varepsilon_n \ll 1$ are positive real constants such that both families are tame.

[The tameness condition is open: At degenerate critical points the Hessian vanishes, which happens only for $\lambda = 0$ and $\alpha = 0$. In the first case the critical points and critical values of $f_0$ coincide with those of $f_0|_{x_1=0}$, so they are in bijection if the latter is a Morse function, which is an open condition. In the analogous second case it suffices to see that $g_0$ restricted to the $x_1$-axis is the Morse function $x_1^{l_1} - x_1$.]

**Proof.** By Theorem 3.1 the versal braid monodromy of the unfolding $\lambda, \alpha, u \mapsto f - \lambda x_1 - \alpha \sum_{i>1} \varepsilon_i x_i$ is equal to the braid monodromy of $f$. So it suffices to show that Proposition 2.2 applies.

Since the bifurcation set in the $\lambda, \alpha$ parameter plane is quasi-homogeneous and contains both axes, we may deduce by the method of Zariski and van Kampen that the fundamental group of the complement is generated by paths in a line parallel to the $\lambda$-axis and a path which is geometric for the $\lambda$-axis. Since such paths lie in the base of the family $f_\lambda$ respectively $g_\alpha$ we are done.

**References**