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Probability Theory

Closedness results for BMO semi-martingales and application to quadratic BSDEs

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Abstract

We give a closedness result for a convex set of BMO semi-martingales, that contains solutions to quadratic BSDEs. We deduce convergence and monotone stability results for quadratic BSDEs. ***To cite this article: P. Barrieu et al., C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Un résultat de fermeture pour des semi-martingales BMO et une application aux EDSRs à croissance quadratique. Nous donnons un résultat de fermeture pour un ensemble convexe de semi-martingales BMO, qui inclut les solutions de EDSRs à croissance quadratique. Nous en déduisons des résultats de convergence et de stabilité monotone pour les EDSRs à croissance quadratique. ***Pour citer cet article : P. Barrieu et al., C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Version française abrégée

L'étude des solutions des équations différentielles rétrogrades (EDSRs) à croissance quadratique met particulièrement en évidence l'intêret des semi-martingales BMO. Plus précisément, nous nous intéressons ici aux EDSRs du type $-\mathrm{d}Y_t = g(t, Y_t, Z_t)\,\mathrm{d}t - Z_t\,\mathrm{d}W_t;\ Y_T = \xi_T \in \mathbb{L}^\infty$ où W est un mouvement Brownien d-dimensionnel défini sur un espace de probabilité filtré $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\leqslant T})$, lorsque le triplet (g, Y_t, Z_t) satisfait :

$$\left|g(\omega, t, y, z)\right| \leqslant c_l + a|y| + \frac{h}{2}|z|^2, \quad \mathrm{d}\mathbb{P}\otimes \mathrm{d}t\text{-p.s.} \tag{1}$$

Toutefois, nous adoptons un cadre d'étude BMO et non le cadre plus standard $\mathbb{H}^2$, i.e. la solution à cette équation est un couple de processus adaptés (Y, Z) tel que Y est un processus continu et borné et la partie martingale $M^Z =$

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$\int_0^{\cdot} Z_s \, \mathrm{d}W_s$ est une martingale BMO. En particulier, nous montrons l'équivalence suivante : $Y_t \in \mathbb{L}^\infty \Leftrightarrow Z_t \in \mathrm{BMO}(\mathbb{P})$ (Proposition 1.1).

Comme conséquence d'un résultat de fermeture pour les martingales exponentielles de martingales BMO (Théorème 2.2), un résultat général de stabilité mixte pour certaines classes de semi-martingales uniformément bornées définies comme $Y_t^n = \mathbb{E}[L_{t,T}^n \xi_T^n + \int_t^T L_{t,s}^n k_s^n \, \mathrm{d}s / \mathcal{F}_t]$ où L^n est une suite de martingales exponentielles associées à des martingales BMO, les processus ξ^n et k^n satisfont certaines propriétés de bornitude, est ensuite obtenu (Proposition 2.3). La version différentielle est une EDSR linéaire dont la classe, notée $\mathcal{S}_{c,k,r}$, est un ensemble convexe, où c, k, r représentent les diverses constantes. Il est également possible de réécrire, sous certaines conditions, une semi-martingale quadratique rétrograde Y définie comme $-\mathrm{d}Y_t = g_t \, \mathrm{d}t - Z_t \, \mathrm{d}W_t;\ Y_T = \xi_T \in \mathbb{L}^\infty$, lorsque le triplet (g, Y, Z) satisfait la condition :

$$|g_t| \leqslant c_l + a|Y_t| + \frac{h}{2}|Z_t|^2, \quad \mathrm{d}\mathbb{P} \otimes \mathrm{d}t\text{-a.s.} \tag{2}$$

de telle sorte que Y soit aussi dans $\mathcal{S}_{c,k,r}$ (Lemma 3.1).

Enfin, dans la dernière partie de cette Note, nous montrons un résultat général de convergence pour les semi-martingales quadratiques satisfaisant la condition (2). Un résultat d'existence pour les EDSRs quadratiques (Théorème 3.3) est obtenu alors sous des hypothèses supplémentaires d'approximation monotone du coefficient g de l'EDSR.

1. BMO-martingales and quadratic BSDEs

Backward Stochastic Differential equations (BSDEs) are equations of the following type:

$$-\mathrm{d}Y_t = g(t, Y_t, Z_t) \, \mathrm{d}t - Z_t \, \mathrm{d}W_t, \quad Y_T = \xi_T \in \mathbb{L}^\infty, \tag{3}$$

where W is a d-dimensional Brownian motion on a filtered space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \leqslant T})$ and (Y, Z) are two adapted processes in the appropriate spaces. Here and after $Z_t \, \mathrm{d}W_t$ simply denotes the scalar product and, when working with BSDEs, the filtration $(\mathcal{F}_t)_{t \leqslant T}$ refers to the natural filtration of the Brownian motion augmented by the $\mathbb{P}$-null sets of $\mathcal{F}$.

Such equations were introduced by Peng and Pardoux in 1990 [8] when the coefficient g is Lipschitz continuous. They were soon recognized as powerful tools. Recently, quadratic BSDEs have recently received an accrued interest. The existence and uniqueness issues for solutions to these quadratic equations, first examined by Kobylanski [6], remain however delicate.

We adopt here a new approach to study these questions. First, the coefficient g of the BSDE (3), defined on the space $\Omega \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^{1 \times d}$ and progressively measurable w.r to (ω, t) satisfies some quadratic growth condition

$$\big|g(\omega, t, y, z)\big| \leqslant c_l + a|y| + \frac{h}{2}|z|^2, \quad \mathrm{d}\mathbb{P} \otimes \mathrm{d}t\text{-a.s.} \tag{4}$$

Second, we consider the BMO-framework instead of the more standard $\mathbb{H}^2$-framework: the solution is an adapted pair of processes (Y, Z) such that Y is a real continuous bounded process and the martingale part $M^Z = \int_0^{\cdot} Z_s \, \mathrm{d}W_s$ in (3) is a BMO-martingale, i.e.

$$\sup_{t \in [0,T]} \mathbb{E}\left[\int_t^T |Z_s|^2 \, \mathrm{d}s / \mathcal{F}_t\right] \text{ is bounded by the so-called norm } \|Z\|_{\mathrm{BMO}(\mathbb{P})}^2.$$

In this Note, we emphasize the flexibility offered by BMO-martingales, especially when dealing with changes of probability measures. In particular, generalizing bounded martingales, BMO-martingales allow a nice extension of Girsanov theorem (see Kazamaki [5, Theorem 3.6]). Several authors have underlined the particular role played by BMO-martingales in the study of quadratic growth BSDEs. Hu, Imkeller and Müller [4] were among the first to use properties of BMO martingales with applications in mathematical finance. The BMO framework appears all the more natural so since it is deeply linked to the quadratic assumption on the generator g, as shown in the proposition below:

Proposition 1.1. *Let (Y_t, Z_t) be a solution in the $(\mathbb{L}^\infty, \mathbb{H}^2)$-framework of*

$$-\mathrm{d}Y_t = g(t, Y_t, Z_t) \, \mathrm{d}t - Z_t \, \mathrm{d}W_t, \quad Y_T = \xi_T \in \mathbb{L}^\infty,$$

such that g satisfies Condition (4). *Then*:

$$Y_T^* = \sup_{0\leqslant t\leqslant T} |Y_t| \in \mathbb{L}^\infty \quad \Leftrightarrow \quad M^Z := Z.W \in \mathrm{BMO}(\mathbb{P}).$$

Besides, $\|Y\|_\infty$ *and* $\|Z\|_{\mathrm{BMO}(\mathbb{P})}$ *are uniformly bounded by constants depending only on a, h and* $\|\xi\|_\infty$.

The proof of this result is rather standard and is omitted here.

2. Mixed stability results for some class of semi-martingales

This section aims at studying some stability results for some class of semi-martingales. Taking the limit of stochastic integrals is often a crucial issue. Several results have been obtained for sequences of martingales (see for instance the survey paper of Delbaen and Schachermayer [2]). Here, we consider this question for BMO-semimartingales, i.e. semimartingales with a BMO-martingale part. We first study some properties of exponential martingales of BMO-martingales with BMO-norm bounded by r. The sets are respectively denoted by $\mathcal{L}_r = \{\mathcal{E}(M),\ M \in \mathrm{BMO}(r)\}$ and $\mathrm{BMO}(\mathbb{P}, r) = \{M \in \mathrm{BMO},\ \|M\|_{\mathrm{BMO}} \leqslant r\}$.

2.1. Closedness theorem of exponential martingales of BMO-martingales

The following theorem presents some key results in the characterization of BMO-martingales and their exponentials (Theorems 2.4 and 3.1. in Kazamaki [5] and Proposition 1 in Doléans-Dade and Meyer [3]):

Theorem 2.1. (i) *If* $M \in \mathrm{BMO}(r)$*, then there exists* $q_r > 1$*, simply depending on* r*, such that* $\mathcal{E}(M)$ *satisfies the following inequality for all stopping times* τ:

$$\mathbb{E}\big[\mathcal{E}(M)_\infty^{q_r}/\mathcal{F}_\tau\big] \leqslant C_{q_r}\mathcal{E}(M)_\tau^{q_r}, \quad \text{a.s.} \tag{5}$$

(ii) $M \in \mathrm{BMO}(r)$ *if and only if for any arbitrary bounded positive martingale* $Y_t = \mathbb{E}[Y_\infty/\mathcal{F}_t]$*,* $\mathcal{E}(M)$ *satisfies the following inequality for all stopping times* τ *and some* $p_r > 1$ *simply depending on* r:

$$\mathcal{E}(M)_\tau Y_\tau^{p_r} \leqslant K_r \mathbb{E}\big[\mathcal{E}(M)_\infty Y_\infty^{p_r}/\mathcal{F}_\tau\big], \quad \text{a.s.} \tag{6}$$

Note that from (i), $\mathcal{E}(M)$ is uniformly integrable and uniformly bounded in $\mathbb{L}^{q_r}$, and $\mathcal{L}_r \subset \mathbb{H}^{q_r}$ for some $q_r > 1$.

A classical closedness result for exponential martingales is that of Yor [9], stating that the limit of a sequence of exponential martingales bounded in $\mathbb{H}^q$, with $q > 1$, is also an exponential martingale in $\mathbb{H}^q$. The previous theorem ensures that the conditions needed to apply Yor's results are satisfied in $\mathcal{L}_r$:

Theorem 2.2. *The set* $\mathcal{L}_r$ *is convex and closed for the convergence in probability of the terminal variables* $\mathcal{E}(M)_\infty$.

Proof. In the sequel, for the sake of simplicity, we also use the generic notation $L_t \equiv \mathcal{E}(M)_t$ for the exponential martingale. The proof of this closedness result is mainly based upon the linearity of the previous Inequality (6). More precisely,

• *Closedness*: Let us consider a sequence L^n of exponential martingales $\mathcal{E}(M^n)$ associated with BMO(r)-martingales M^n, such that their terminal values L_∞^n converge in probability towards L_∞^*. We want to prove that the associated process L^* is in fact an element of $\mathcal{L}_r$. As previously emphasized, thanks to Inequality (5), the sequence L_∞^n is uniformly integrable in $\mathbb{L}^{q'}$ for $1 < q' < q_r$. L_∞^n is a bounded sequence in $\mathbb{L}^{q_r}$, that converges uniformly in $\mathbb{L}^{q'}$ towards L_∞^*. Hence the sequence of martingales L_t^n converges uniformly in t in $\mathbb{H}^{q'}$ towards L_t^*, and from Yor [9], we know that L^* is an exponential martingale $\mathcal{E}(M^*)$ of a local martingale M^*. From Inequality (6), which is asymptotically stable, M^* is in fact a true martingale of BMO(r).

• *Convexity*: Let us consider a convex combination α of exponential martingales L^i, $\bar{L}_t^\alpha \equiv \sum_i \alpha_i L_t^i$. Therefore, $\mathrm{d}\bar{L}_t^\alpha/\bar{L}_t^\alpha = (\sum_i \alpha_i L_t^i\,\mathrm{d}M_t^i)/\sum_i \alpha_i L_t^i = \sum_i \hat{\alpha}_{i,t}\,\mathrm{d}M_t^i \equiv \mathrm{d}\hat{M}_t$, where $\hat{\alpha}_{i,t} = \alpha_i L_t^i/(\sum_j \alpha_j L_t^j)$. The next step is then to prove that $\hat{M}$ is a BMO(r)-martingale. This is a direct consequence of Inequality (6), which ensures that any convex combination of L^i also satisfies Inequality (6) and therefore, from Theorem 2.1, the associated martingale is in BMO(r). □

2.2. Mixed stability result for some semi-martingale class

Let us first introduce some useful notation on sequences of convex combinations: The direct convex combinations of X^i, for $i \geqslant n$, are denoted as $\bar{X}^{\alpha,n} = \sum_{i \geqslant n} \alpha_i^n X^i$. For random convex combinations, i.e. combinations with random weights $\hat{\alpha}_{i,t}$ defined as above, we use the notation $\hat{X}^n$.

We now consider a family of uniformly bounded semi-martingales $Y_t^n = \mathbb{E}[L_{t,T}^n \xi_T^n + \int_t^T L_{t,s}^n k_s^n \, \mathrm{d}s / \mathcal{F}_t]$, where (ξ_T^n) is a sequence of uniformly bounded $\mathcal{F}_T$-measurable random variables, (k^n) is a sequence of uniformly bounded $\mathcal{F}_t$-adapted processes, $L^n = \mathcal{E}(M^n)$ is a sequence of exponential martingales of $\mathcal{L}_r$ and $L_{t,T}^n = L_T^n / L_t^n$.

Proposition 2.3. (i) *There exists a sequence of convex combination* (α^n) *such that* $\hat{\xi}_T^{\alpha,n}$ *and* $\hat{k}_t^{\alpha,n}$ *converges in* $\mathbb{L}^p$ *towards* $\hat{\xi}_T^*$ *and* $\hat{k}_t^*$ *respectively and such that the exponential martingale* $\bar{L}_t^{\alpha,n}$ *converges towards* $\bar{L}_t^*$ *in* $\mathbb{H}^{q'}$, *for* $1 < q' < q_r$ *and* $\frac{1}{p} + \frac{1}{q'} < 1$. *Moreover,* $\hat{Y}_t^{\alpha,n} = \mathbb{E}[\bar{L}_{t,T}^{\alpha,n} \hat{\xi}_T^{\alpha,n} + \int_t^T \bar{L}_{t,s}^{\alpha,n} \hat{k}_s^{\alpha,n} \, \mathrm{d}s / \mathcal{F}_t]$ *converges uniformly in* $\mathbb{H}^p$ *towards* $\hat{Y}_t^* = \mathbb{E}[\bar{L}_{t,T}^* \hat{\xi}_T^* + \int_t^T \bar{L}_{t,s}^* \hat{k}_s^* \, \mathrm{d}s / \mathcal{F}_t]$.

(ii) *If* Y_τ^n *converges towards* Y_τ *in probability for all stopping times* $\tau \leqslant T$, *then* $Y_\tau = \hat{Y}_\tau^*$.

Proof. As a result of Theorem 2.2, the exponential martingale $\bar{L}^{\alpha,n}$ converges to the exponential martingale $\bar{L}^*$ in $\mathbb{H}^{q'}$. Moreover, since $\hat{\xi}_T^{\alpha,n}$ is uniformly bounded, there exists a convex combination, still denoted by $\hat{\xi}_T^{\alpha,n}$, converging in any $\mathbb{L}^p$-space towards ξ_T^*. The same holds true for $\hat{k}^{\alpha,n}$. The convergence of $\hat{Y}_t^{\alpha,n} = \mathbb{E}[\bar{L}_{t,T}^{\alpha,n} \hat{\xi}_T^{\alpha,n} / \mathcal{F}_t]$ is uniform, as a consequence of both the convergence of terminal values and the theorem of martingale convergence. If in addition, the sequence (Y^n) converges, then the convex combination $\hat{Y}^\alpha$ converges towards the same limit. Hence the results. □

2.3. Link with linear BSDEs

The class of semimartingales studied above is strongly connected with the class of following linear BSDEs. More precisely, let (Y, Z) be a solution of the linear BSDE: $-\mathrm{d}Y_t = k_t \, \mathrm{d}t - Z_t(\mathrm{d}W_t + \Theta_t \, \mathrm{d}t)$; $Y_T = \xi_T$.

When Y is uniformly bounded by c, k_t by k, and Θ_t is in $\mathrm{BMO}(\mathbb{P}, r)$, Y is said to be in $\mathcal{S}_{c,k,r}$.

Proposition 2.4. *The class* $\mathcal{S}_{c,k,r}$ *is convex and closed w.r to the convergence in probability of the solutions* Y.

Both Propositions 2.4 and 2.3 are identical after having observed that the dual representation of such a linear BSDE is: $Y_t = \mathbb{E}[L_{t,T} \xi_T + \int_t^T L_{t,s} k_s \, \mathrm{d}s / \mathcal{F}_t]$, where $L = \mathcal{E}(-\Theta \cdot W)$. Note also that as a consequence of Proposition 1.1, since Y is bounded, Z is in $\mathrm{BMO}(\mathbb{P})$, with a BMO-norm that only depends on the constants c, k and r. The coefficient of this linear BSDE is $g(t, y, z) = k_t - z\theta$. While the standard framework for linear BSDEs involves a bounded process Θ and a terminal condition in $\mathbb{L}^2$, here for a bounded terminal condition, we are able to reach BMO processes Θ.

Remark. Our study has some connection with some existing results. In particular, the closedness properties in $\mathbb{L}^2$ of BMO-semi-martingale $Y^n = Z^n \cdot X$ with $\mathrm{d}X_t = \mathrm{d}W_t + \Theta \, \mathrm{d}t$ have been established by Delbaen et al. in [1]. Our approach is however different since we study stability results of BMO-semi-martingales $Y^n = Z^n \cdot X^n$ with $\mathrm{d}X_t^n = \mathrm{d}W_t + \Theta^n \, \mathrm{d}t$ under some structural conditions of the type uniform BMO on the Θ^n. Our results coincide when Θ_t is given but the main point of this study is thus to use the previous results on exponential martingale of BMO-martingales to obtain the convergence, up to some convex combination, of the Θ^n sequence.

3. Stability results for quadratic backward semi-martingales

In this section, we study quadratic backward semi-martingales Y defined as $-\mathrm{d}Y_t = g_t \, \mathrm{d}t - Z_t \, \mathrm{d}W_t$; $Y_T = \xi_T \in \mathbb{L}^\infty$, such that Y is uniformly bounded by c, Z is in BMO and the triplet (g, Y, Z) satisfies the following condition:

$$|g_t| \leqslant c_l + a|Y_t| + \frac{h}{2}|Z_t|^2, \quad \mathrm{d}\mathbb{P} \otimes \mathrm{d}t\text{-a.s.} \tag{7}$$

Note that Proposition 1.1 ensures that the BMO-norm of Z is uniformly bounded by a constant c_z that simply depends on c_l a, h and c.

3.1. Quadratic semi-martingales and the class $\mathcal{S}_{c,k,r}$

Before presenting some stability results, we relate these quadratic semi-martingales with the class $\mathcal{S}_{c,k,r}$ of linear BSDEs previously introduced. The following algebraic transformation of the coefficient g will be useful in the following to obtain some stability results:

Lemma 3.1. *Under Condition* (7)*, the quadratic backward semi-martingale Y defined as $-\mathrm{d}Y_t = g_t\,\mathrm{d}t - Z_t\,\mathrm{d}W_t$; $Y_T = \xi_T \in \mathbb{L}^\infty$ belongs to the class $\mathcal{S}_{c,k,r}$, where c is the uniform bound of Y, $k = c_l + ac$ and r is $\frac{h^2}{4}c_z$.*

Proof. From Condition (7) and given the assumptions on Y, there exists a constant k such that $c_l + a|Y_t| \leqslant k$. Hence, denoting $g_t \equiv f_t + \frac{h}{2}|Z_t|^2$, we have $-k - h|Z_t|^2 \leqslant f_t \leqslant k$. The idea is then to write f_t as: $f_t = (f_t + k)^+ - (f_t + k)^- - k$. The process $k_t = (f_t + k)^+ - k$ is bounded by k.

Hence $0 \leqslant (f_t + k)^- \leqslant h|Z_t|^2$ and there exists a process U_t such that $0 \leqslant U_t \leqslant 1$ and $(f_t + k)^- = hU_t|Z_t|^2$. Coming back to the g_t decomposition, we introduce the process $\Theta_t \equiv h(U_t - \frac{1}{2})Z_t$, and rewrite g_t as $g_t = k_t - Z_t\Theta_t$. Since Z is in $\mathrm{BMO}(\mathbb{P})$, with a uniform BMO norm by assumption and U is uniformly bounded by construction, Θ is also in $\mathrm{BMO}(\mathbb{P})$. We denote this BMO-norm by r. Hence the result. □

3.2. Monotone stability of quadratic semi-martingales

The following theorem gives a key result on the convergence of sequences of quadratic backward semi-martingales. The proof of this result is a straightforward alternative to standard proofs that can be found in the literature (e.g. Kobylanski [6]).

Theorem 3.2. *Let us consider a uniformly bounded sequence of quadratic backward semi-martingales*

$$-\mathrm{d}Y_t^n = g_t^n\,\mathrm{d}t - Z_t^n\,\mathrm{d}W_t; \quad Y_T^n = \xi_T \in \mathbb{L}^\infty,$$

such that Condition (7) *is satisfied.*

1. *Let us assume that Y^n converges almost surely uniformly towards a process Y.*
   (i) *The limit process Y is in the class $\mathcal{S}_{c,k,r}$ with the representation $-\mathrm{d}Y_t = k_t^*\,\mathrm{d}t - Z_t^*(\mathrm{d}W_t + \Theta_t^*\,\mathrm{d}t)$.*
   (ii) *The sequence Z^n is a Cauchy sequence for the BMO-norm converging towards the process Z^*.*
2. *If the sequence Y^n converges monotonically almost surely towards a process Y, then Y is a continuous process and the convergence is uniform.*

Proof. 1. (i) The result on Y is obtained as a straightforward application of Proposition 2.3.

(ii) From Condition (7), since the process Z^n is in $\mathrm{BMO}(\mathbb{P}, r)$, we have, for any i, j and any $u \in [0, T]$, $\mathbb{E}[\int_u^T |g_s^i - g_s^j|\,\mathrm{d}s/\mathcal{F}_u] \leqslant \mathbb{E}[\int_u^T (|g_s^i| + |g_s^j|)\,\mathrm{d}s/\mathcal{F}_u] \leqslant C_g$, where C_g is related to the BMO constant of Z^n. This inequality is the key argument to prove the convergence of the Z^n in BMO. More precisely, $|Y_t^i - Y_t^j|^2 + \mathbb{E}[\int_t^T |Z_s^i - Z_s^j|^2\,\mathrm{d}s/\mathcal{F}_t] \leqslant 2\mathbb{E}[\int_t^T |Y_s^i - Y_s^j||g_s^i - g_s^j|\,\mathrm{d}s/\mathcal{F}_t] \leqslant 2\mathbb{E}[\int_t^T \sup_{t\leqslant u\leqslant s}|Y_u^i - Y_u^j||g_s^i - g_s^j|\,\mathrm{d}s/\mathcal{F}_t]$.

Denoting the increasing process $\sup_{t\leqslant u\leqslant s}|Y_u^i - Y_u^j|$ by $A_{t,s}^{i,j}$ and using an integration by part formula, we can rewrite $\mathbb{E}[\int_t^T (A_{t,s}^{i,j} - A_{t,t}^{i,j})|g_s^i - g_s^j|\,\mathrm{d}s/\mathcal{F}_t] = \mathbb{E}[\int_t^T \mathrm{d}A_{t,u}^{i,j}\mathbb{E}[\int_u^T |g_s^i - g_s^j|\,\mathrm{d}s/\mathcal{F}_u]/\mathcal{F}_t]$.

Using the inequality on g^i previously noticed,

$$\begin{aligned}
\left|Y_t^i - Y_t^j\right|^2 + \mathbb{E}\left[\int_t^T \left|Z_s^i - Z_s^j\right|^2\mathrm{d}s/\mathcal{F}_t\right] &\leqslant 2C_g\left|Y_t^i - Y_t^j\right| + 2C_g\mathbb{E}\left[A_{t,T}^{i,j} - A_{t,t}^{i,j}/\mathcal{F}_t\right] \\
&\leqslant 2C_g\mathbb{E}\Big[\sup_{t\leqslant u\leqslant T}\left|Y_u^i - Y_u^j\right|/\mathcal{F}_t\Big].
\end{aligned}$$

Finally, $\mathbb{E}[\int_t^T |Z_s^i - Z_s^j|^2 \, ds/\mathcal{F}_t] \leqslant 2C_g \mathbb{E}[\sup_{0\leqslant u\leqslant T} |Y_u^i - Y_u^j|/\mathcal{F}_t]$. From the a.s. uniform convergence of Y^i, (Z^i) is a BMO(r) Cauchy sequence whose the limit is also in BMO(r). Note that this result is obtained without any particular knowledge on the convergence of the sequence of coefficients g^n.

2. comes from Dini's Theorem. □

3.3. Existence of quadratic BSDEs

Using the previous results, we now prove the existence of a minimal solution for the quadratic BSDEs $-dY_t = g(t, Y_t, Z_t)\,dt - Z_t\,dW_t$, when the coefficient g satisfies Condition (4). More precisely, we approximate the coefficient g by a monotone sequence g^n. Both g and g^n are supposed to be continuous. Therefore the convergence of g^n to g is uniform on all compact sets.

Theorem 3.3. *We consider an increasing sequence of continuous functions g^n defined as*:

$$g^n(t, y, z) = g(t, y, z) \vee \left(-c_l + ac - hn|z| + \frac{h}{2}|z|^2\right).$$

(i) *There exists a minimal solution (Y^n, Z^n) in $\mathbb{L}^\infty \times$ BMO to the BSDE $-dY_t^n = g^n(t, Y_t^n, Z_t^n)\,dt - Z_t^n\,dW_t$ and the sequence Y^n is non-decreasing.*

(ii) *There exists a minimal solution (Y, Z) in $\mathbb{L}^\infty \times$ BMO to the BSDE $-dY_t = g(t, Y_t, Z_t)\,dt - Z_t\,dW_t$.*

Proof. The idea is to write the function g^n as $g^n(t, y, z) = g(t, y, z) \vee (-c_l + ac - hn|z| + \frac{h}{2}|z|^2) = f^n(t, y, z) + \frac{h}{2}|z|^2$, where f^n is continuous with linear growth in (y, z). Using the standard exponential transformation, we can rewrite the problem in terms of a BSDE with continuous coefficients having a linear growth in (y, z). The results from Lepeltier and San Martin [7] ensure the existence of a minimal solution to the BSDE associated with g^n and the sequence Y^n is non-decreasing. □

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