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C. R. Acad. Sci. Paris, Ser. I 346 (2008) 825–828

COMPTES RENDUS MATHEMATIQUE

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Complex Analysis

An extremal problem for a class of entire functions

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Received 29 May 2008; accepted 18 June 2008

Available online 25 July 2008

Presented by Paul Malliavin

Abstract

Let f be an entire function of the exponential type, such that the indicator diagram is in $[-\mathrm{i}\sigma, \mathrm{i}\sigma]$, $\sigma > 0$. Then the upper density of f is bounded by $c\sigma$, where $c \approx 1.508879$ is the unique solution of the equation

$$\log\left(\sqrt{c^2+1}+c\right) = \sqrt{1+c^{-2}}.$$

This bound is optimal. ***To cite this article: A. Eremenko, P. Yuditskii, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Un problème extrêmal pour une classe de fonctions entières. Soit f une fonction entière d'indicatrice contenue dans l'intervalle $[-\mathrm{i}\sigma, \mathrm{i}\sigma]$, $\sigma > 0$. Alors la borne supérieure des zéros de f ne dépasse pas $c\sigma$, où $c \approx 1{,}508879$ est la solution d'équation,

$$\log\left(\sqrt{c^2+1}+c\right) = \sqrt{1+c^{-2}}.$$

Cette borne est exacte. ***Pour citer cet article : A. Eremenko, P. Yuditskii, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

We consider the class E_σ, $\sigma > 0$ of entire functions of exponential type whose indicator diagram is contained in a segment $[-\mathrm{i}\sigma, \mathrm{i}\sigma]$, which means that

$$h(\theta) := \limsup_{r\to+\infty} \frac{\log|f(r\mathrm{e}^{\mathrm{i}\theta})|}{r} \leqslant \sigma|\sin\theta|, \quad |\theta| \leqslant \pi. \tag{1}$$

An alternative characterization of such functions follows from a theorem of Pólya [6]:

$$f(z) = \frac{1}{2\pi}\int_\gamma F(\zeta)\mathrm{e}^{-\mathrm{i}\zeta z}\,\mathrm{d}\zeta,$$

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[1] Supported by NSF grant DMS-0555279.
[2] Supported by Austrian Fund FWF P20413-N18.

1631-073X/$ – see front matter
doi:10.1016/j.crma.2008.06.009

where F is an analytic function in $\bar{\mathbf{C}} \setminus [-\sigma, \sigma]$, $F(\infty) = 0$, and γ is a closed contour going once around the segment $[-\sigma, \sigma]$. In other words, the class of entire functions satisfying (1) are Fourier transforms of hyperfunctions supported by $[-\sigma, \sigma]$, see, for example, [2] and [3].

Let $n(r)$ be the number of zeros of f in the disc $\{z\colon\ |z| \leqslant r\}$, counting multiplicity. We are interested in the *upper density*:

$$D = \limsup_{r\to\infty} \frac{n(r)}{r}. \tag{2}$$

If f satisfies the additional condition:

$$\int_{-\infty}^{\infty} \frac{\log^+ |f(x)|}{1+x^2}\, \mathrm{d}x < \infty, \tag{3}$$

then the limit (density) in (2) exists and equals $(2\pi)^{-1} \int_{-\pi}^{\pi} h(\theta)\, \mathrm{d}\theta$. For example, if $f(z) = \sin \sigma z$, then $f \in E_\sigma$ and $D = 2\sigma/\pi \approx 0.6366\sigma$. The existence of the limit follows from a theorem of Levinson [5,6]. Much more precise information about $n(r)$ under the condition (3) is contained in the theorem of Beurling and Malliavin [1].

In the general case, the density might not exist as was shown by examples in [4,10]. Moreover, it is possible that $D > 2\sigma/\pi$, see [2]. An easy estimate using Jensen's formula gives $D \leqslant 2\mathrm{e}\sigma/\pi \approx 1.7305\sigma$. This estimate is exact in the larger class of entire functions satisfying the condition $h(\theta) \leqslant \sigma$, but it is not exact in E_σ.

In this Note we find the best possible upper estimate for the upper density of zeros of functions in E_σ.

Theorem. *The upper density of zeros of a function $f \in E_\sigma$ does not exceed $c\sigma$, where $c \approx 1.508879$ is the unique solution of the equation*:

$$\log\bigl(\sqrt{c^2+1} + c\bigr) = \sqrt{1+c^{-2}}, \quad \text{on } (0, +\infty). \tag{4}$$

For every $\sigma > 0$ there exist entire functions $f \in E_\sigma$ such that $D = c\sigma$.

Proof. Without loss of generality we assume that $\sigma = 1$. Moreover, it is enough to consider only even functions. To make a function f even we replace it by $f(z)f(-z)$, which results in multiplication of both the indicator h and the upper density D by the same factor of 2.

Let $t_n \to +\infty$ be such sequence that $\lim n(t_n)/t_n = D$. Consider the sequence of subharmonic functions $v_n(z) = t_n^{-1} \log |f(t_n z)|$. Compactness Principle for subharmonic functions [3, Theorem 4.1.9] implies that one can choose a subsequence that converges in $\mathcal{D}'$ (Schwartz's distributions). The limit function v is subharmonic in the plane, and satisfies:

$$v(z) \leqslant |\mathrm{Im}\, z|, \quad z \in \mathbf{C}, \quad \text{and} \quad v(0) = 0. \tag{5}$$

Let μ be the Riesz measure of this function. We have to show that

$$\mu\bigl(\{z\colon\ |z| \leqslant 1\}\bigr) \leqslant c. \tag{6}$$

First we reduce the problem to the case that the Riesz measure μ is supported by the real line. We have

$$v(z) = \frac{1}{2} \int \log\left|1 - \frac{z^2}{\zeta^2}\right| \mathrm{d}\mu_\zeta.$$

Let us compare this with

$$v^*(z) = \frac{1}{2} \int_0^\infty \log\left|1 - \frac{z^2}{t^2}\right| \mathrm{d}\mu^*_t,$$

where μ^* is the radial projection of the measure μ: it is supported on $[0, +\infty)$ and $\mu^*(a,b) = \mu(\{z\colon\ a < |z| < b\})$, $0 \leqslant a < b$. It is easy to see that

$$v^*(z) \leqslant \sigma' |\mathrm{Im}\, z|, \quad z \in \mathbf{C}, \quad \text{and} \quad v^*(0) = 0, \tag{7}$$

with some $\sigma' > 0$. We claim that one can choose $\sigma' \leqslant 1$ in (7). Let σ' be the smallest number for which (7) holds. Then, by the subharmonic version of the theorem of Levinson mentioned above (see, for example, [9]), the limit

$$\lim_{r\to\infty} r^{-1}v^*(rz) = \sigma'|\operatorname{Im} z|$$

exists in $\mathcal{D}'$ and thus

$$\lim_{r\to\infty} \frac{1}{r}\int_0^r \frac{n_{v^*}(t)}{t}\,\mathrm{d}t = \lim_{r\to\infty} \frac{1}{2\pi r}\int_{-\pi}^{\pi} v^*\bigl(r\mathrm{e}^{\mathrm{i}\theta}\bigr)\,\mathrm{d}\theta = \frac{2\sigma'}{\pi},$$

where

$$n_{v^*}(r) = \mu^*[0,r] = \mu\bigl\{z\colon\ |z| \leqslant r\bigr\}. \tag{8}$$

Similar limits exist for v, and we have $n_v = n_{v^*}$, from which we conclude that $\sigma' \leqslant 1$.

From now on we assume that v is harmonic in the upper and lower half-planes, and that

$$v(\mathrm{i}y) \sim y, \quad y \to +\infty. \tag{9}$$

Let u be the harmonic function in the upper half-plane such that $\phi = u + iv$ is analytic, and $\phi(0) = 0$. Then ϕ is a conformal map of the upper half-plane onto some region G of the form:

$$G = \bigl\{x + \mathrm{i}y\colon\ y > g(x)\bigr\}, \tag{10}$$

where g is an even upper semi-continuous function, $g(0) = 0$. Moreover,

$$\phi(\mathrm{i}y) \sim \mathrm{i}y, \quad \text{as } y \to +\infty, \tag{11}$$

which follows from (9), and

$$\phi(-\bar{z}) = -\overline{\phi(z)}, \tag{12}$$

because both the region G and the normalization of ϕ are symmetric with respect to the imaginary axis. Finally, we have:

$$\mu\bigl([0,x]\bigr) = \frac{2}{\pi}u(x). \tag{13}$$

For all these facts we refer to [7,8].

Remark. The function $\operatorname{Re}\phi(x) = u(x)$ might be discontinuous for $x \in \mathbf{R}$. We agree to understand $u(x)$ as the limit from the right $u(x+0)$ which always exists since u is increasing.

Inequality (5) implies that $v(x) \leqslant 0$, thus $g(x) \leqslant 0$, in other words, G contains the upper half-plane.

Thus we obtain the following extremal problem: *Among all univalent analytic functions ϕ satisfying* (12) *and mapping the upper half-plane onto regions of the form* (10) *with $g \leqslant 0$, $g(0) = 0$ and satisfying $\phi(0) = 0$ and* (11)*, maximize* $\operatorname{Re}\phi(1)$.

We claim that the extremal function g for this problem is:

$$g_0(x) = \begin{cases} -\infty, & 0 < |x| < \pi c/2, \\ 0, & \text{otherwise,} \end{cases}$$

where $c > 1$ is the solution of Eq. (4). The corresponding region is shown in Fig. 1. For the extremal function we have $\phi_0(1) = \pi c/2 - \mathrm{i}\infty$.

To prove the claim, we first notice that for a given G the mapping function is uniquely defined. Let $a = \phi(1)$, and $b = \operatorname{Re} a$. Next we show that making g smaller on the interval $(0,b)$ results in increasing $\operatorname{Re}\phi(1)$ and for g larger on the interval $(b,+\infty)$ also results in increasing $\operatorname{Re}\phi(1)$. The proofs of both statements are similar. Suppose that $g_1 \leqslant g$, $g_1 \neq g$, and $g_1(x) = g(x)$ outside of the two intervals $p < |x| < q$, where $0 < p < q < b$. Let G_1 be the

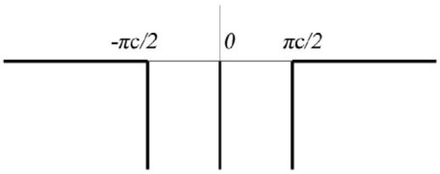

Fig. 1. Extremal region.

region above the graph of g_1, and ϕ_1 the corresponding mapping function normalized in the same way as g. Then $G \subset G_1$, and the conformal map $\phi_1^{-1} \circ \phi$ is defined in the upper half-plane and maps it into itself. We have:

$$\phi_1^{-1} \circ \phi(x) = x + 2x \int_0^\infty \frac{w(t)}{t^2 - x^2} \,\mathrm{d}t,$$

where $w \neq 0$ is a non-negative function supported on some interval inside (0, 1). Putting $x = 1$ we obtain:

$$\phi_1^{-1}(a) = 1 + 2 \int_0^\infty \frac{w(t)}{t^2 - 1} \,\mathrm{d}t,$$

so $\phi_1^{-1}(a) < 1$, that is $\operatorname{Re} \phi_1(1) > b$. This proves our claim.

It remains to compute the constant b in the extremal domain. We recall that $\phi_0(1) = b - \mathrm{i}\infty$ and assume that $b = \phi_0(k)$ for some $k > 1$. Here ϕ_0 is the extremal mapping function. Then by the Schwarz–Christoffel formula we have:

$$\phi_0(z) = \frac{1}{2} \int_0^{z^2} \frac{\sqrt{\zeta - k^2}}{\zeta - 1} \,\mathrm{d}\zeta. \tag{14}$$

To find k, we use the condition that

$$\operatorname{Im} p.v. \int_0^{k^2} \frac{\sqrt{\zeta - k^2}}{\zeta - 1} \,\mathrm{d}\zeta = 0.$$

Denoting $c = \sqrt{k^2 - 1}$ and evaluating the integral, we obtain

$$\log\left(\sqrt{c^2 + 1} + c\right) = \sqrt{1 + c^{-2}}.$$

Finally the jump of the real part of the integral in (14) occurs at the point 1 and has magnitude $\pi\sqrt{k^2 - 1} = \pi c$. This completes the proof of the upper estimate in Theorem 1.

To construct an example showing that this estimate can be attained, we follow the construction in [2, Sections 9–10]. The role of the subharmonic function u_1 there is played now by our extremal function $v_0 = \operatorname{Im} \phi_0$. $\square$

References

[1] A. Beurling, P. Malliavin, On Fourier transforms of measures with compact support, Acta Math. 118 (1967) 291–309.
[2] A. Eremenko, D. Novikov, Oscillation of Fourier integrals with a spectral gap, J. Math. Pures Appl. 83 (3) (2004) 313–365.
[3] L. Hörmander, Analysis of Linear Partial Differential Operators, vols. I, II, Springer, Berlin, 1983.
[4] P. Kahane, L. Rubel, On Weierstrass products of zero type on the real axis, Illinois Math. J. 4 (1960) 584–592.
[5] P. Koosis, Leçons sur le théorème de Beurling et Malliavin, Publ. CRM, Montréal, 1996.
[6] B. Levin, Distribution of Zeros of Entire Functions, Amer. Math. Soc., Providence, RI, 1980.
[7] B. Levin, Subharmonic majorants and some applications, in: Complex Analysis, Birkhäuser, Basel, 1988, pp. 181–190.
[8] B. Levin, The connection of a majorant with a conformal mapping, II, Teor. Funktsii Funktional Anal. i Prilozhen. 52 (1989) 3–21 (in Russian). English translation in: J. Soviet Math. 52 (5) (1990) 3351–3364.
[9] V. Matsaev, M. Sodin, Distribution of Hilbert transforms of measures, Geom. Funct. Anal. 10 (2000) 1, 160–184.
[10] C. Roumieu, Sur quelques extensions de la notion de distribution, Ann. Sci. École Norm. Sup. 77 (1960) 41–121.