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Dynamical Systems

The boundary of bounded polynomial Fatou components

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Abstract

We prove that, for a polynomial, every bounded Fatou component, with the exception of Siegel disks, has for boundary a Jordan curve. ***To cite this article: P. Roesch, Y. Yin, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Frontière des composantes de Fatou polynômiales. Nous montrons que le bord de toute composante de Fatou bornée d' un polynôme, hormis les disques de Siegel, est une courbe de Jordan. ***Pour citer cet article : P. Roesch, Y. Yin, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Theorem 1. *Let B be a bounded Fatou component of a polynomial f that is not eventually a Siegel disk, then ∂B is a Jordan curve.*

By hypothesis, B is the connected component of the immediate basin of attraction of an (eventually) attracting or parabolic periodic point. For simplicity of the exposition, we assume that the Julia set is connected and we write the proof in the attracting case, but the arguments are the one of the parabolic (and attracting) case. We fix a point x on ∂B and construct a basis of connected neighborhoods as follows. Let Γ be a graph formed by an internal and an external equipotential, a cycle of internal rays generated by some $R_B(\xi)$ (where ξ is periodic such that $\bigcup_n f^n(x) \cap \overline{R_B(\xi)} = \emptyset$) together with the external rays landing at their terminal points on ∂B. We denote by $P_n(*)$, the puzzle pieces, which are the connected components of $\mathbb{C} \setminus f^{-n}(\Gamma)$. In the parabolic case internal rays (resp. equipotentials) are parabolic rays (resp. parabolic equipotentials) see [4] or [6]. The puzzle piece containing x, $P_n(x)$, is well defined and $\overline{P_n(x)} \cap \partial B$ is a connected set since it is the intersection of the compact connected sets consisting in $\bar{\phi}(\Delta_n)$ where $\phi: \mathbf{D} \to B$ is the Riemann map and $\Delta_n = \{re^{it} \mid r \in [1 - 1/n[,\ t \in [t_1, t_2]\}$ some sector. Thus, let $\mathrm{Imp}(x)$ be the intersection $\bigcap_n \overline{P_n(x)}$. If $\mathrm{Imp}(x) \cap \partial B = \{x\}$, the sequence $(P_n(x))$ forms the investigated basis. Note that it does not depend on the graph. Then Theorem 1 follows since by Carathéodory's Theorem if ∂B is locally connected, it is a curve and it is simple by the maximum principle.

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Proposition 2. *If x is eventually periodic, either* $\mathrm{Imp}(x)=\{x\}$ *or two external rays converge to x separating* $\mathrm{Imp}(x)\setminus\{x\}$ *from B so that* $\mathrm{Imp}(x)\cap\partial B=\{x\}$.

Denote by *Crit* the set of critical points of f. We assume that for any z, $P_0(z)\cap Crit\subset\mathrm{Imp}(z)$ (up to starting the puzzle at a deeper level).

Proof. The proof uses an argumentation similar to the one of Kiwi (see [2]). We assume (up to replacing f by an iterate) that x is a fixed point and we suppose that $\mathrm{Imp}(x)\neq\{x\}$. The accessible fixed points in $\mathrm{Imp}(x)$ give us fixed points of a map of $\mathbb{S}^1$ as follows: Since $\mathrm{Imp}(x)$ is a compact connected set (as intersection of such sets), we can consider a Riemann map $\phi:\overline{\mathbb{C}}\setminus\mathrm{Imp}(x)\to\overline{\mathbb{C}}\setminus\overline{\mathbb{D}}$. The map $g=\phi\circ f\circ\phi^{-1}$ is well defined on $\phi(P_0(x)\setminus\mathrm{Imp}(x))$ since $(P_0(x)\setminus\mathrm{Imp}(x))\cap Crit=\emptyset$ so that there is no pre-image of $\mathrm{Imp}(x)$ in $P_0(x)$ other than $\mathrm{Imp}(x)$. First of all, if $\mathrm{Imp}(x)\cap\partial P_0(x)\neq\emptyset$ (this happens only in the parabolic case), we consider U_0 a small enlargement of $P_0(x)$ at those points still satisfying the previous properties; then let $V_0=\phi(U_0\setminus\mathrm{Imp}(x))$, $U_1=f^{-1}(U_0\setminus\mathrm{Imp}(x))$ and $V_1=\phi(U_1)$. By Schwarz reflection principle on V_1 and V_0, we get $\tilde V_1$ and $\tilde V_0$ neighborhoods of $\mathbb{S}^1$ and a map $\tilde g:\tilde V_1\to\tilde V_0$ such that $\tilde g_{|V_1}=g$. Since $\tilde g$ is a holomorphic covering that preserves $\mathbb{S}^1$ and each side, it has no critical point on $\mathbb{S}^1$.

We now consider particular fixed points coming from rays in $P_n(x)$ as follows. In $\partial P_n(x)\cap B$, there are portions of two internal rays $R_B(t_n)$, $R_B(t'_n)$ and portions of two external rays $R(\eta_n)$, $R(\eta'_n)$ landing at the previous landing points. The angles t_n,t'_n, respectively η_n,η'_n, converge to some t, respectively to η,η', since the puzzle pieces are nested. Moreover, $R_B(t)$, $R(\eta)$, $R(\eta')$ are fixed rays, since $f(P_n(x))=P_{n-1}(x)$. Let y be the landing point of $R_B(t)$ and z,z' of $R(\eta)$, $R(\eta')$. Since y is either repelling or parabolic, it is the landing point of an external ray, say $R(\theta)$, which is fixed since $R_B(t)$ is a fixed internal ray. The curve $\phi(R(\theta))$ converges to some point in $\mathbb{S}^1$ and we chose ϕ such that this point is 1. We suppose to get a contradiction that $y\neq z$ (the same argument holds for $y\neq z'$). Then the limit of $\phi(R(\eta))$ and 1 are two fixed points of $\tilde g$. It is not very difficult to see that the fixed points of $\tilde g$ are all weakly repelling, that is $|\tilde g(x)-p|_{\mathbb{S}^1}>|x-p|_{\mathbb{S}^1}$ for a fixed point p (see [2,6]). Then, it is easy to see on a lift of $\tilde g$ that between two fixed points of $\tilde g$ there is a preimage by $\tilde g$ of 1. Thus, each connected component of $V_1\setminus(\phi(R(\theta))\cup\phi(R(\eta)))$ contains a preimage of $\phi(R(\theta))$. The same holds for the open set $U_1\setminus\mathrm{Imp}(x)$: it is cut in two components by $R(\theta)$ and $R(\eta)$ and each one contains a preimage of $R(\theta)$. Hence there are $R(\theta')$, $R(\theta'')$ preimages of $R(\theta)$ such that $\theta<\theta'<\eta<\theta''<\theta+2\pi$. Since $R(\theta')$, $R(\theta'')$ land on $\mathrm{Imp}(x)$ these rays enter every puzzle piece $P_n(x)$ so that at least one of the angle θ',θ'' belongs to the intervals (η_n,η), (η'_n,η') whose diameters tend to 0. This gives the contradiction since it would implies that θ' or θ'' equal η or η'. Then $y=z=z'$ so that $R(\eta)$ and $R(\eta')$ land at the same point.

Let W_0 be the union of the connected components of $P_0(x)\setminus(R(\eta)\cup R(\eta')\cup\{z\})$ intersecting B. Assume to get a contradiction that $Imp(x)\cap W_0=I\neq\emptyset$. The set I contains more than one point. Either there is a critical point in I so that there is a preimage of y in I with a preimage $R(\zeta)$ of $R(\theta)$ landing at it or the map $f:W_1\to W_0$ is an homeomorphism where W_1 is the connected component of $P_1(x)\setminus(\overline{R}(\eta)\cup\overline{R}(\eta'))$ intersecting B and $W_1\subset W_0$. In the first case the angle ζ belongs to one of the intervals $(\eta_n,\eta)\cup(\eta'_n,\eta')$ whose diameters tend to 0, thus $\zeta=\eta$ which gives the contradiction. In the second case, $f_0^{-1}:W_0\to W_1$ is a conformal map. By Denjoy–Wolff's Theorem, the iterates by f_0^{-1} of every point of W_0 converge to a unique point in $\overline{W}_0$, but this is not possible since $I\subset(f_0^{-1})^n(W_0)$ for all $n>0$. Hence the two rays $R(\eta)$, $R(\eta')$ separate $\overline{B}$ from $\mathrm{Imp}(x)\setminus\{x\}$. Finally, the sets $\overline{P}_n(x)\cap\partial B$ form a basis of connected neighborhoods of x in ∂B. □

Lemma 3. *If x is not eventually periodic but the ω-limit set of x contains an eventually periodic point y, then* $\mathrm{Imp}(x)\cap\partial B=\{x\}$.

Note that any point of $\omega(x)$ is *accumulated* by x in the following sense:

Definition 4. We say that z_1 *accumulates* z_2 if for every n there exists k such that $f^k(z_1)\in P_n(z_2)$.

Denote by b, resp. *dmax*, the number of critical impressions, resp. the maximal degree of f on those sets. To prove Lemma 3 we will use the following lemma:

Lemma 5. *For $k \geqslant 0$ and for t the smallest integer such that $f^t(x) \in P_k(c)$ for some c, the degree of $f^t : P_{k+t}(x) \to P_k(c)$ is bounded by $dmax^b$.*

Proof of Lemma 5. Every critical point appears at most once in the sequence $f^i(P_{k+t}(x))$ for $0 \leqslant i \leqslant t$, otherwise c would appear in some puzzle piece $P_k(f^j(x))$ for $j < t$. The result follows. □

Proof of Lemma 3. We assume (up to iterate f) that y is fixed. It is on ∂B and by Proposition 2 $\mathrm{Imp}(y) \cap \partial B = \{y\}$. Here we choose a graph for y, that will be also good for x. Hence, for every $n \in \mathbb{N}$ there exists a smallest k_n such that $f^{k_n}(x) \in P_n(y)$ and a smallest $m_n \geqslant n$ such that $f^{k_n}(x) \notin P_{m_n}(y)$. Hence $y \in P_{m_n-1-i}(f^{k_n+i}(x)) \setminus P_{m_n-i}(f^{k_n+i}(x))$ for every $i \leqslant m_n - 1$. So the map $f^{m_n} : P_{m_n}(f^{k_n}(x)) \to P_0(f^{m_n+k_n}(x))$ is an homeomorphism, since $P_0(y) \cap Crit \subset \mathrm{Imp}(y)$. The annulus $A = P_0(y) \setminus \overline{P}_1(y)$ is non degenerate for any graph that do not contain the internal fixed ray converging to y. So let $k'_n \geqslant m_n + k_n + 1$ be the smallest such that $f^{k'_n}(x) \in P_1(y)$. Then the modulus of $P_{k'_n}(x) \setminus \overline{P}_{k'_n+1}(x)$ is bigger than or equal to $\mathrm{mod}(A)/dmax^{2b}$. Indeed, the map $f^{m_n+k_n} : P_{m_n+k_n}(x) \to P_0(f^{m_n+k_n}(x))$ has degree less than or equal to $dmax^b$ since k_n is the first integer such that $f^{k_n}(x) \in P_n(y)$, and the degree of $f^{k'_n-(m_n+k_n)} : P_{k'_n-(m_n+k_n)}(f^{m_n+k_n}(x)) \to P_0(f^{k'_n}(x))$ is less or equal than $dmax^b$. Now, choose $n_1 \geqslant k'_n$ and repeat the argument again, and so on. One gets a sequence of annuli $P_{k'_{n_i}}(x) \setminus \overline{P}_{k'_{n_i}+1}(x)$ of modulus bounded below by $\mathrm{mod}(A)/dmax^{2b}$. Therefore $\mathrm{Imp}(x) = \{x\}$. □

Definition 6. Denote by $Crit(z)$ the set of critical points accumulated by a point z. Let $Crit_a$ be the set of critical points accumulating themselves and $Crit_{nr}$ the ones not accumulating points of $Crit$. They are disjoint sets and every critical point belongs to or accumulates on $Crit_a \cup Crit_{nr}$ (since the property of accumulation is transitive). So we concentrate on $Crit_a$ that we decompose in $Crit_{rr} \cup Crit_{pr}$ as follows. A piece $P_{n+k}(c')$ is a *child* of $P_n(c)$ if $c, c' \in Crit$, c accumulates c' and $f^{k-1} : P_{n+k-1}(f(c')) \to P_n(c)$ is conformal. Then c is said *persistently recurrent*, i.e. in $Crit_{pr}$, if for every $c' \in Crit(c)$ accumulating c, $P_n(c')$ has a finite number of children for every $n \geqslant 0$. Else c is said *reluctantly recurrent*, in $Crit_{rr}$.

In the following we consider only points that do not contain eventually periodic points in their ω-limit set:

Proposition 7. *If $Crit(x) = \emptyset$ or if $Crit(x) \cap (Crit_{nr} \cup Crit_{rr}) \neq \emptyset$, then* $\mathrm{Imp}(x) = \{x\}$.

Lemma 8. *For any puzzle piece P_{n_0} and any sequence (n_k), such that $f^{n_k}(x) \in P_{n_0}$, there exists some $r \geqslant 1$ and a subsequence (n_{k_i}) of (n_k) such that $f^{n_{k_i}}(x) \in P_{n_0+r}$ and $\overline{P}_{n_0+r} \subset P_{n_0}$.*

Proof. In this proof, we consider only puzzle pieces containing infinitely many points of the sequence $f^{n_k}(x)$. Either there is a puzzle piece P_{n_0+r} compactly contained in P_{n_0} for some $r \geqslant 1$ or, for every decreasing sequence P_{n_0+r}, ∂P_{n_0+r} intersects $\partial P_{n_0} \cap \partial B$ at one point $= \{z\}$ for every $r \geqslant 1$. The point z is eventually repelling since ∂P_{n_0} and ∂P_{n_0+1} can only intersect at the landing points of rays. Therefore, $\mathrm{Imp}(z) \cap \partial B = \{z\}$ (by Proposition 2) so that $z \in \omega(x)$ since $x \in \partial B$. This contradicts the hypothesis that there is no eventually periodic point in $\omega(x)$. □

Corollary 9. *If P_{n_0} and (n_k) satisfy that $f^{n_k} : P_{n_k+n_0}(x) \to P_{n_0}$ has bounded degree, then* $\mathrm{Imp}(x) = \{x\}$.

Proof. By Lemma 8, for this puzzle piece P_{n_0} and this sequence (n_k), we can find a puzzle piece P_{n_0+r} such that $\overline{P}_{n_0+r} \subset P_{n_0}$ and $f^{n_{k_i}}(x) \in P_{n_0+r}$ for a subsequence n_{k_i} of n_k. Then by classical results, the modulus $\mathrm{mod}(P_{n_0+n_{k_i}}(x) \setminus \overline{P}_{n_0+r+n_{k_i}}(x))$ is greater than $\mathrm{mod}(P_{n_0} \setminus \overline{P}_{n_0+r})/D$ where D is the bound on the degree. Therefore $\mathrm{Imp}(x) = \{x\}$.

Corollary 9 implies that we do not need to search non degenerate annuli in this special case. □

Proof of Proposition 7. If $Crit(x) = \emptyset$, there is a level n_0 such that there is no critical points in $P_{n_0+j}(f^i(x))$ for $i, j > 0$. Then for every $k \geqslant 0$ the map $f^k : P_{n_0+k}(x) \to P_{n_0}(f^k(x))$ is of degree at most $dmax$. Since there is only a finite number of puzzle pieces of depth n_0, there are a puzzle piece P_{n_0} and a subsequence $\{k_i\}$ such that $P_{n_0}(f^{k_i}(x)) = P_{n_0}$. The result follows from Corollary 9.

If $c \in Crit(x) \cap Crit_{nr}$, there is a level n_0 such that for every $k \geqslant 0$ the map $f^k : P_{n_0+k}(c) \to P_{n_0}(f^k(c))$ is of degree at most *dmax*. Let t_k be the smallest integer such that $f^{t_k}(x) \in P_{n_0+k}(c)$. By Lemma 5 the degree of $f^{t_k} : P_{n_0+k+t_k}(x) \to P_{n_0+k}(c)$ is bounded by $dmax^b$. There exists at least one piece P_{n_0} such that $P_{n_0} = P_{n_0}(f^k(c))$ for infinitely many values of $k \geqslant 0$. Therefore the assumptions of Corollary 9 are satisfied and the result follows.

If x accumulates $c \in Crit_{rr}$, there exists $c', c'' \in Crit$, a level n_0 and a sequence (k_i) such that the map $f^{k_i} : P_{n_0+k_i}(c'') \to P_{n_0}(c')$ is of degree at most *dmax* (there are infinitely many children of $P_{n_0}(c')$). For every $i \geqslant 0$ let t_i be the smallest integer such that $f^{t_i}(c) \in P_{n_0+k_i}(c'')$ and let l_i be the smallest integer such that $f^{l_i}(x) \in P_{n_0+k_i+l_i}(c)$, they exist since c accumulates c'' and x accumulates c. By Lemma 5 the degree of $f^{l_i+t_i+k_i} : P_{n_0+l_i+t_i+k_i}(x) \to P_{n_0}(c')$ is bounded by $dmax^{2b+1}$. As above the result follows. $\square$

We assume now that $c_0 \in Crit(x) \subset Crit_{pr}$. For any puzzle piece P_{n_0} containing c_0 Lemma 8 gives for some $r \geqslant 1$ a non degenerate annulus $P_{n_0} \setminus \overline{P}_{n_0+r}$. We construct from P_{n_0+r} a nest (K_n), (K'_n) called KSS nest (see [3]) as follows. Define the first hit containing a point z to a puzzle piece I as $L_z(I)$ the connected component containing z of the points of I coming back in I for the first time. Thus, to any puzzle piece I containing c_0, we associate two puzzle pieces $A(I)$ and $B(I)$ satisfying the following properties: $\overline{A(I)} \subset B(I)$, for some t the map $f^t : B(I) \to I$ has degree less than $dmax^{b^2}$, the orbit of critical points accumulated by c_0 never meets $B(I) \setminus A(I)$ and $f^t(A(I)) = L_{f^t(c_0)}(I)$. Then consider K_0, K'_0 defined by $K_0 = B(A(I))$ with $f^{t_0}(K_0) = A(I)$ and K'_0 is the connected component containing c_0 of $(f^{t_0})^{-1}(B(I))$. Let us continue the induction by taking $I_1 = \Gamma^T(K_0)$ where T is some constant and $\Gamma(J)$ is the last successor of J. The last successor of a puzzle piece J is a puzzle piece $\Gamma(J)$ such that $f^j(\Gamma(J)) = J$ with the largest $j = l + m + o$ such that $f^o(\Gamma(J)) = Q_{c'}$ is a child of some P_c (exactly $f^m(Q_{c'}) = P_c$) and $f^l(P_c) = J$. In particular we have the property that $f^i(\Gamma(J))$ does not contain c_0 for all $0 < i < j$. Iterating the process, we obtain a sequence I_n and then the KSS nest (K_n), (K'_n).

Lemma 10. *Let (K_n), (K'_n) be the KSS nest constructed from P_{n_0+r}, then* $\mathrm{level}(K'_n) - \mathrm{level}(K_n) \to \infty$.

Proof. Denote by $r(P)$ the minimal time of first return to P of points of P. By definition, $\mathrm{level}(K'_n) - level(K_n) = \mathrm{level}(B(I_n)) - \mathrm{level}(A(I_n)) \geqslant r(I_n)$. Now, for J, j as above, as $f^i(\Gamma(J))$ does not contain c_0 for all $0 < i < j$, $r(\Gamma(J)) \geqslant q$ where $f^q(\Gamma(J)) = J$. Since every piece has at least two successors, $r(\Gamma(J)) \geqslant 2r(J)$. Therefore $r(I_{n+1}) \geqslant 2^T r(I_n)$ so that $r(I_n)$ tends to ∞ and so does $\mathrm{level}(K'_n) - \mathrm{level}(K_n)$. $\square$

Corollary 11. *The set $K'_n \setminus \overline{K}_n$ is a non degenerate annulus for large n.*

Proof. For j_n such that $f^{j_n}(K_n) = P_{n_0+r}$, the annulus $K'_n \setminus \overline{K}_n$ contains $P_{n_0+j_n} \setminus \overline{P}_{n_0+j_n+r}$ for large n (Lemma 10). So it is non degenerate since $f^{j_n}(P_{n_0+j_n}) = P_{n_0}$ and $f^{j_n}(P_{n_0+j_n+r}) = P_{n_0+r}$.

Proposition 12. *If every c accumulated by $x \in \partial B$ belongs to $Crit_{pr}$, then* $\mathrm{Imp}(x) = \{x\}$.

Proof. Since the annuli $K'_n \setminus \overline{K}_n$ are non degenerate, the same argument as in [5] allows us to conclude that there is some positive constant μ such that $\mathrm{mod}(K'_n \setminus \overline{K}_n) \geqslant \mu$ (this uses in particular Kahn–Lyubich's Lemma, see [1]). The pieces K_n, K'_n are of the form $P_{k_n}(c_0)$, $P_{k'_n}(c_0)$. Since x accumulates c_0, let l_n be the smallest integer such that $f^{l_n}(x) \in K_n$. By Lemma 5, the degree of the map $f^{l_n} : P_{k_n+l_n}(x) \to P_{k_n}(c_0)$ is bounded by $dmax^b$. Moreover, the orbit of any the critical point accumulated by c_0 does never enter $K'_n \setminus K_n$, by the construction of the KSS nest. Therefore the degree is bounded by the same constant D for the map $f^{l_n} : P_{k'_n+l_n}(x) \to P_{k'_n}(c_0)$ so that the modulus of the annulus $P_{k'_n+l_n}(x) \setminus \overline{P}_{k_n+l_n}(x)$ is greater than μ/D which implies that $\mathrm{Imp}(x) = \{x\}$. $\square$

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