Available online at www.sciencedirect.com
ScienceDirect
C. R. Acad. Sci. Paris, Ser. I 346 (2008) 749–752
COMPTES RENDUS MATHEMATIQUE
http://france.elsevier.com/direct/CRASS1/

Functional Analysis

The Szegő and Avram–Parter theorems for general test functions

Albrecht Böttcher [a], Sergei M. Grudsky [b], Egor A. Maksimenko [c]

[a] *Fakultät für Mathematik, TU Chemnitz, 09107 Chemnitz, Germany*
[b] *Departamento de Matemáticas, CINVESTAV del I.P.N., Apartado Postal 14-740, 07000 México, D.F., Mexico*
[c] *Department of Mathematics, Mechanics and Computer Science, Southern Federal University, 344006 Rostov-on-Don, Russia*

Received 26 October 2007; accepted after revision 3 June 2008

Available online 24 June 2008

Presented by Yves Meyer

Abstract

The Szegő and Avram–Parter theorems give the limit of the arithmetic mean of the values of certain test functions at the eigenvalues of Hermitian Toeplitz matrices and the singular values of arbitrary Toeplitz matrices, respectively, as the matrix dimension goes to infinity. We show that, surprisingly, these theorems are not true for every continuous, nonnegative, and monotonously increasing test function and thus do not hold whenever they make sense. On the other hand, we prove the two theorems in a general form which includes all versions known so far. ***To cite this article: A. Böttcher et al., C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Les théorèmes de Szegő et d'Avram–Parter pour des fonctions test générales. Les théorèmes de Szegő et d'Avram–Parter donnent la limite de la moyenne arithmetique des valeurs d'une 'bonne' fonction test prise en les valeurs propres de matrices de Toeplitz hermitiennes et en les valeurs singulières de matrices de Toeplitz arbitraires quand la dimension de la matrice tend vers l'infini. Nous montrons que, de manière surprenante, ces théorèmes ne sont pas valables pour une fonction test continue, positive et croissante arbitraire, alors même que leur énoncé a bien un sens. En revanche, nous prouvons les deux théorémes sous une forme générale qui inclut toutes les versions connues jusqu'ici. ***Pour citer cet article : A. Böttcher et al., C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

The $n \times n$ Toeplitz matrix $T_n(a)$ generated by a complex-valued function a in $L^1 := L^1(0, 2\pi)$ is the matrix $(a_{j-k})_{j,k=1}^n$ where $a_\ell = \int_0^{2\pi} a(\theta) \mathrm{e}^{-\mathrm{i}\ell\theta} \frac{\mathrm{d}\theta}{2\pi}$ is the ℓth Fourier coefficient of a. If a is real-valued, then the matrices $T_n(a)$ are all Hermitian and theorems of the Szegő type say that, under certain conditions,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} G\bigl(\lambda_j^{(n)}\bigr) = \int_0^{2\pi} G\bigl(a(\theta)\bigr) \frac{\mathrm{d}\theta}{2\pi}, \tag{1}$$

E-mail addresses: aboettch@mathematik.tu-chemnitz.de (A. Böttcher), grudsky@math.cinvestav.mx (S.M. Grudsky), emaximen@mail.ru (E.A. Maksimenko).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2008.06.002

where $\lambda_1^{(n)} \leqslant \cdots \leqslant \lambda_n^{(n)}$ are the eigenvalues of $T_n(a)$. Theorems of the Avram–Parter type do not require that a be real-valued. They state that, again under appropriate assumptions,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} F\big(s_j^{(n)}\big) = \int_0^{2\pi} F\big(|a(\theta)|\big) \frac{\mathrm{d}\theta}{2\pi}, \tag{2}$$

where, this time, $s_1^{(n)} \leqslant \cdots \leqslant s_n^{(n)}$ is the singular values of $T_n(a)$, that is, the nonnegative square roots of the eigenvalues of $T_n(\bar{a})T_n(a)$.

The functions G and F in (1) and (2) are referred to as test functions, and we will always assume that G is nonnegative and continuous on $\mathbf{R}$, $G \in C_+(\mathbf{R})$, and that F is nonnegative and continuous on $[0,\infty)$, $F \in C_+[0,\infty)$. Serra Capizzano [4] solved the problem of characterizing the maximal class of test functions for which (1) and (2) hold under the assumption that both sides of (1) and (2) are finite. He showed that this happens for all $a \in L^p$ $(p \geqslant 1)$ if and only if $G(\lambda) = \mathrm{O}(|\lambda|^p)$ as $|\lambda| \to \infty$ and $F(s) = \mathrm{O}(s^p)$ as $s \to \infty$. Here, we also admit the case where both sides of (1) and (2) are infinite. We denote the functions under the integrals in (1) and (2) by $G(a)$ and $F(|a|)$, respectively. If these functions are not in L^1, the integral is given the value $+\infty$.

Let ST denote the set of all $G \in C_+(\mathbf{R})$ for which (1) holds for all real-valued $a \in L^1$ and let APT be the set of all $F \in C_+[0,\infty)$ such that (2) is valid for all $a \in L^1$. As larger and larger subsets of ST and APT have been identified over the years (see [2] and [4]), we are led to the question whether the Szegő and Avram–Parter theorems are true whenever they make sense, which is equivalent to the question whether $ST = C_+(\mathbf{R})$ and $APT = C_+[0,\infty)$. Note that Golinskii and Ibragimov [3] proved that the so-called strong Szegő limit theorem for positive generating functions is indeed true whenever it makes sense (see also the books [2] and [6]). In [1] we showed that the answer to our question is negative: ST and APT are proper subsets of $C_+(\mathbf{R})$ and $C_+[0,\infty)$, respectively.

The counterexample in [1] is highly oscillating and leaves us with the question whether ST and APT contain at least all monotonous functions. Our first main result tells us that, surprisingly, this is not the case:

Theorem 1. *There exist monotonously increasing C^∞ functions in $C_+(\mathbf{R}) \setminus ST$ and $C_+[0,\infty) \setminus APT$.*

The proof of this theorem is rather sophisticated. It is based on the following construction. For $k \geqslant 1$, put $b_k = \exp(\exp k^2)$, $\beta_k = 1/b_k$, $\delta_k = \exp(-k^2)$. Let G be any monotonously increasing C^∞ function on $\mathbf{R}$ which is identically zero on $(-\infty, 0]$, takes the value b_k on $[b_{k-1}+1, b_k]$ $(k \geqslant 2)$, and increases linearly on $[b_k + \beta_k, b_k + 1 - \beta_k]$ $(k \geqslant 1)$. Denote by F the restriction of G to $[0,\infty)$. Define $a \in L^1$ by $a(\theta) = b_k$ for $\theta \in [(1-\delta_k)\beta_k, \beta_k] =: I_k$ $(k \geqslant 1)$ and $a(\theta) = 0$ for $\theta \in [0, 2\pi) \setminus (I_1 \cup I_2 \cup \cdots)$. It is easily seen that in the case at hand the right-hand sides of (1) and (2) are finite. We can prove that the upper limit of $(1/n)\sum G(\lambda_j^{(n)}) = (1/n)\sum F(s_j^{(n)})$ is infinite, which implies that $G \notin ST$ and $F \notin APT$.

So which functions belong to ST and APT? Zamarashkin and Tyrtyshnikov [8] showed that all compactly supported continuous functions are in ST and APT, Tilli [7] proved that the same is true if only uniform continuity is required, and Serra Capizzano [4] observed that F is in APT if F is in $C_+[0,\infty)$ and $F(s) = \mathrm{O}(s)$ as $s \to \infty$. Here is our second main result:

Theorem 2. *The following are equivalent*:

(i) *APT contains all compactly supported functions in $C_+[0,\infty)$*;
(ii) *APT contains all monotonously increasing and convex functions in $C_+[0,\infty)$.*

Since (i) is known to be true from [8], we arrive at the conclusion that (ii) is also true. Our proof of Theorem 2 is based on a variational characterization of the sums $\sum F(s_j^{(n)})$ for monotonously increasing and convex functions F which mimics the variational characterization of unitarily invariant matrix norms given in [5]. To be more precise, we can prove the following:

Lemma 3. *Let $F \in C_+[0,\infty)$ and $n \geqslant 2$. Then the following are equivalent*:

(i) *F is monotonously increasing and convex*;

(ii) *for every matrix* $A \in \mathbf{C}^{n\times n}$ *we have*

$$\sum_{j=1}^{n} F\big(s_j^{(n)}\big) = \max \sum_{k=1}^{n} F\big(\big|\langle Au_k, v_k\rangle\big|\big),$$

where $s_1^{(n)} \leqslant \cdots \leqslant s_n^{(n)}$ *are the singular values of* A *and the maximum is over all pairs* $\{u_1, \ldots, u_n\}$ *and* $\{v_1, \ldots, v_n\}$ *of orthonormal bases of* $\mathbf{C}^n$ *with the standard inner product* $\langle\cdot,\cdot\rangle$.

Using Lemma 3 we get the following inequality, which for $F(s) = s^p$ $(1 \leqslant p < \infty)$ was already established in [4] and [5] and which plays a key role in the proof of Theorem 2:

Lemma 4. *If* $a \in L^1$ *and if* $F \in C_+[0,\infty)$ *is monotonously increasing and convex, then*

$$\frac{1}{n}\sum_{j=1}^{n} F\big(s_j^{(n)}\big) \leqslant \int_0^{2\pi} F\big(\big|a(\theta)\big|\big)\,\frac{\mathrm{d}\theta}{2\pi}$$

for all $n \geqslant 1$.

Theorem 2, in conjunction with additional analysis, yields the following two theorems. For $\Phi(s) = Cs^p$ $(1 \leqslant p < \infty)$, part (a) of Theorem 5 is due to Serra Capizzano [4]. Our proof of Theorem 5 is based on ideas of [4] as well.

Theorem 5.

(a) *Let* $a \in L^1$, *let* $\Phi \in C_+[0,\infty)$ *be a monotonously increasing and convex function, and suppose* $\Phi(|a|) \in L^1$. *If* $F \in C_+[0,\infty)$ *is any function such that* $F(s) \leqslant \Phi(s)$ *for all sufficiently large* $s > 0$, *then* (2) *holds.*
(b) *Let* $a \in L^1$ *be real-valued, put* $a_+ = \max(a,0)$, $a_- = \max(-a,0)$, *let* $\Phi_\pm \in C_+[0,\infty)$ *be monotonously increasing and convex functions such that* $\Phi_-(0) = \Phi_+(0)$, *and suppose* $\Phi_+(a_+)$ *and* $\Phi_-(a_-)$ *are in* L^1. *If* $G \in C_+(\mathbf{R})$ *satisfies* $G(\lambda) \leqslant \Phi_+(\lambda)$ *and* $G(-\lambda) \leqslant \Phi_-(\lambda)$ *for all sufficiently large* $\lambda > 0$, *then* (1) *is valid.*

We write $H(x) \simeq \Phi(x)$ as $x \to \infty$ if there exist two positive constants c_1, c_2 such that $c_1\Phi(x) \leqslant H(x) \leqslant c_2\Phi(x)$ for all sufficiently large $x > 0$.

Theorem 6.

(a) *If* $F \in C_+[0,\infty)$ *and* $F(s) \simeq \Phi(s)$ *as* $s \to \infty$ *for some convex function* $\Phi \in C_+[0,\infty)$, *then* $F \in APT$.
(b) *If* $G \in C_+(\mathbf{R})$ *and if there exist two convex functions* $\Phi_\pm \in C_+[0,\infty)$ *such that* $G(\lambda) \simeq \Phi_+(\lambda)$ *as* $\lambda \to \infty$ *and* $G(-\lambda) \simeq \Phi_-(\lambda)$ *as* $\lambda \to \infty$, *then* $G \in ST$.

In short, the Szegő and Avram–Parter theorems are always true for essentially convex test functions. Note that monotonicity is no longer required.

Acknowledgements

The second author acknowledges financial support by CONACYT grant 046936-F.

References

[1] A. Böttcher, S.M. Grudsky, M. Schwartz, Some problems concerning the test functions in the Szegő and Avram–Parter theorems, Operator Theory: Advances and Applications, in press.
[2] A. Böttcher, B. Silbermann, Analysis of Toeplitz Operators, second ed., Springer, Berlin, 2006.
[3] B.L. Golinskii, I.A. Ibragimov, A limit theorem of G. Szegő, Math. USSR Izvestiya 5 (1971) 421–446.
[4] S. Serra Capizzano, Test functions, growth conditions and Toeplitz matrices, Rend. Circ. Mat. Palermo, Ser. II 68 (Suppl.) (2002) 791–795.

[5] S. Serra Capizzano, P. Tilli, On unitarily invariant norms of matrix-valued linear positive operators, J. Inequal. Appl. 7 (2002) 309–330.
[6] B. Simon, Orthogonal Polynomials on the Unit Circle, Part 1, Classical Theory, Amer. Math. Soc., Providence, RI, 2005.
[7] P. Tilli, A note on the spectral distribution of Toeplitz matrices, Linear and Multilinear Algebra 45 (1998) 147–159.
[8] N.L. Zamarashkin, E.E. Tyrtyshnikov, Distribution of the eigenvalues and singular numbers of Toeplitz matrices under weakened requirements on the generating function, Sb. Math. 188 (1997) 1191–1201.