Differential Geometry

The length of a shortest geodesic loop

Hans-Bert Rademacher

Universität Leipzig, Mathematisches Institut, 04081 Leipzig, Germany

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Abstract

We give a lower bound for the length of a non-trivial geodesic loop on a simply-connected and compact manifold of even dimension with a non-reversible Finsler metric of positive flag curvature. Harris and Paternain use this estimate in their recent paper to give a geometric characterization of dynamically convex Finsler metrics on the 2-sphere. To cite this article: H.-B. Rademacher, C. R. Acad. Sci. Paris, Ser. I 346 (2008).

Résumé


On a compact and simply-connected Riemannian manifold with positive sectional curvature $0 < K \leq 1$ the length of a non-constant geodesic loop is bounded from below by $2\pi$. This result is due to Klingenberg [3] and is of importance in proofs of the classical sphere theorem.

For a compact manifold $M$ with non-reversible Finsler metric $F$ the author introduced in [5] the reversibility

\[ \lambda := \max\{F(−X); F(X) = 1\} \geq 1. \]

In this short Note we show how one can use the results and methods from [5] to obtain the following estimate for the length of a geodesic loop depending on the flag curvature and the reversibility:

**Proposition 1.** Let $M$ be a compact and simply-connected differentiable manifold of even dimension $n \geq 2$ equipped with a non-reversible Finsler metric $F$ and flag curvature $K$ satisfying $0 < K \leq 1$. Then the length $l$ of a shortest non-constant geodesic loop is bounded from below:

\[ l \geq \pi(1 + \lambda^{-1}). \]

In [5, Theorem 4] it is shown that with the same assumptions the length of a closed geodesic $c$ satisfies this estimate. Therefore, Proposition 1 follows from Proposition 3 which we are going to prove in this Note. Proposition 1 answers...
a question posed to the author by G. Paternain. Using results by Hofer, Wysocki and Zehnder [2] and the statement of Proposition 1 Harris and Paternain obtain the following geometric characterization of dynamically convex Finsler metrics on the 2-sphere:

**Proposition 2.** (Harris–Paternain [1, Section 6].) Let $F$ be a non-reversible Finsler metric on the 2-sphere with reversibility $\lambda$ and flag curvature

$$\left(1 - \frac{1}{1+\lambda}\right)^2 < K \leq 1.$$ 

Then the Finsler metric is dynamically convex, in particular there are either two geometrically distinct closed geodesics or there are infinitely many geometrically distinct ones.
**Proof.** Let \( c : [0, l] \to M \) be a shortest geodesic loop parametrized by arc length with \( c(0) = c(l) = p \). Let \( q = c(t), t \in (0, l) \) be the cut point, i.e. \( c|[0, t] \) is minimal. We assume that \( l < \pi (1 + \lambda^{-1}) \). By Lemma 1 we obtain \( 2d < \pi (1 + \lambda^{-1}) \). Then we conclude from Lemma 2 that \( l = L(c) = 2d \). Since

\[
2d = \pi (1 + \lambda^{-1}) = \theta(p, q) + \theta(q, p) = 2d(p, q)
\]

and \( q \in \text{Cut}(p) \) it follows from the definition of the symmetrized injectivity radius \( d \) that equality holds in inequality (1). Therefore \( c[t, l] \) is a minimal geodesic joining \( p \) and \( q \).

For sufficiently small \( \epsilon > 0 \) with \( p\epsilon = c(\epsilon) \) there is \( t\epsilon \in (\epsilon, 2d) \) such that \( q\epsilon = c(t\epsilon) \in \text{Cut}(p\epsilon) \), i.e. the geodesic \( c[\epsilon, t\epsilon] \) is minimal. We conclude from the triangle inequality:

\[
2d(p\epsilon, q\epsilon) = \theta(p\epsilon, q\epsilon) + \theta(q\epsilon, p\epsilon) \leq \theta(p\epsilon, q\epsilon) + \theta(q\epsilon, p) + \theta(p, p\epsilon) = \theta(p, q) + \theta(q, p) = L(c) = 2d(p, q).
\]

From the definition of the symmetrized injectivity radius \( d \) it follows that actually equality holds, i.e. the geodesic loop is a closed geodesic.

**Proof of Proposition 1.** We assume that \( l < \pi (1 + \lambda^{-1}) \) and conclude from Lemmas 1 and 2: \( 2d = \pi (1 + \lambda^{-1}) \). Then Proposition 3 implies that \( L = l = 2d < \pi (1 + \lambda^{-1}) \). But in [5, Theorem 4] it is shown that under the assumptions of Proposition 1 the length \( L \) of a shortest closed geodesic satisfies: \( L \gtrsim \pi (1 + \lambda^{-1}) \). Therefore we obtain a contradiction, i.e. \( l \gtrsim \pi (1 + \lambda^{-1}) \). □

**Remark 1.** Under the assumptions of Proposition 3 we have shown that for any point \( p \in M \) with a cut point \( q \in M \) satisfying \( d(p, q) = d \) there is a shortest closed geodesic \( c : [0, 2d] \to M \) parametrized by arc length passing through \( p \) and \( q \), i.e. \( p = c(0) = c(2d); q = c(t); t \in (0, 2d) \). Hence the restrictions \( c_1 = c|[0, t] \) and \( c_3 = c|[t, 2d] \) are minimal geodesics. The cut point \( q = c(t) \) is not a conjugate point since \( t = \theta(p, q) \lessdot \pi \) and \( K \lessdot 1 \). This implies that there is another minimal geodesic \( c_2 : [0, t] \to M \) joining \( p \) and \( q \). Therefore Proposition 3 excludes the second case discussed in [5, Remark 1] resp. [6, Lemma 9.7(b)].

**References**


