Probability Theory

Error calculus and regularity of Poisson functionals:
the lent particle method

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Abstract

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Résumé


1. Notation and basic formulae

Let us consider a local Dirichlet structure with carré du champ \((X, \mathcal{X}, \nu, d, \gamma)\) where \((X, \mathcal{X}, \nu)\) is a \(\sigma\)-finite measured space called bottom-space. Singletons are in \(X\) and \(\nu\) is diffuse, \(d\) is the domain of the Dirichlet form \(\epsilon[u] = 1/2 \int \gamma[u] \, dv\). We denote \((a, D(a))\) the generator in \(L^2(\nu)\) (cf. [3]).

A random Poisson measure associated to \((X, \mathcal{X}, \nu)\) is denoted \(N\). \(\Omega\) is the configuration space of countable sums of Dirac masses on \(X\) and \(\mathcal{A}\) is the \(\sigma\)-field generated by \(N\), on \(\Omega\). The space \((\Omega, \mathcal{A}, \mathbb{P})\) is called the up-space. We write \(N(f) = \int f \, dN\). If \(p \in [1, \infty]\) the set \(\{e^{i\tilde{N}(f)} : f \text{ real}, f \in L^1 \cap L^2(\nu)\}\) is total in \(L^p(\Omega, \mathcal{A}, \mathbb{P})\). We put \(\tilde{N} = N - \nu\). The relation \(\mathbb{E}(\tilde{N} f)^2 = \int f^2 \, dv\) extends and gives sense to \(\tilde{N}(f), f \in L^2(\nu)\). The Laplace functional and the differential calculus with \(\gamma\) yield

\[
\forall f \in d, \forall h \in D(a) \quad \mathbb{E}\left[e^{i\tilde{N}(f)} \left(\tilde{N}(a[h]) + \frac{i}{2} N(\gamma[f, h])\right)\right] = 0. \tag{1}
\]

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2. Product, particle by particle, of a Poisson random measure by a probability measure

Given a probability space \((R, \mathcal{R}, \rho)\), let us consider a Poisson random measure \(N \circ \rho\) on \((X \times R, \mathcal{X} \times \mathcal{R})\) with intensity \(v \times \rho\) such that for \(f \in L^1(v)\) and \(g \in L^1(\rho)\) if \(N(f) = \sum f(x_n)\) then \((N \circ \rho)(fg) = \sum f(x_n)g(r_n)\) where the \(r_n\)'s are i.i.d. independent of \(N\) with law \(\rho\). Calling \((\hat{\mathcal{O}}, \hat{A}, \hat{\mathbb{P}})\) the product of all the factors \((R, \mathcal{R}, \rho)\) involved in the construction of \(N \circ \rho\), we obtain the following properties: For an \(\mathcal{A} \times \mathcal{X} \times \mathcal{R}\)-measurable and positive function \(F\),

\[
\hat{E}\left(\int F \, d(N \circ \rho)\right)^2 = \int F^2 \, dN \, d\rho \quad \mathbb{P}\text{-a.s.}
\]

Let us denote by \(\mathbb{P}_N\) the measure \(\mathbb{P}(d\omega)N_{\omega}(dx)\) on \((\Omega \times X, \mathcal{A} \times \mathcal{X})\). We have the following:

**Lemma 2.1.** Let \(F\) be \(\mathcal{A} \times \mathcal{X} \times \mathcal{R}\)-measurable, \(F \in L^2(\mathbb{P}_N \times \rho)\) and such that \(\int F(\omega, x, r) \rho(\mathrm{d}r) = 0 \mathbb{P}_N\text{-a.s.},\) then \(\int F \, d(N \circ \rho)\) is well defined, belongs to \(L^2(\mathbb{P} \times \hat{\mathbb{P}})\) and

\[
\hat{E}\left(\int F \, d(N \circ \rho)\right)^2 = \int F^2 \, dN \, d\rho \quad \mathbb{P}\text{-a.s.}
\]

The argument consists in considering \(F_n\) satisfying \(\mathbb{E}\int F_n^2 \, dv \, d\rho < +\infty\) and \(\mathbb{E}\int (|F_n|)^2 \, dv \, d\rho < +\infty\) and using the relation \(\hat{E}(\int F_n(\omega, x, r) \rho(\mathrm{d}r)) = (\int F_n \, dN)\rho) - (\int (F_n \, d\rho)^2 \, dN + \int F_n^2 \, d\rho \, dN \mathbb{P}\text{-a.s.}\)

3. Construction by Friedrichs’ method and expression of the gradient

(a) We suppose the space by \(d\) of the bottom structure is separable, then a gradient exists (cf. [3] Chap. V, p. 225 et seq.). We denote it \(\flat\) and choose it with values in the space \(L^2(\mathcal{V}, \mathcal{A})\). Thus, for \(u \in d\) we have \(u^2 \in L^2(v \times \rho), \gamma[u] = |u|^2 \, d\rho\) and \(\flat\) satisfies the chain rule. We suppose in addition, what is always possible, that \(\flat\) takes its values in the subspace orthogonal to the constant \(1\), i.e.

\[
\forall u \in d, \quad \int u^2 \, d\rho = 0 \quad v\text{-a.s.}
\]

This hypothesis is important here as in many applications (cf. [2] Chap. V §4.6). We suppose also, but this is not essential (cf. [3] p. 44) \(1 \in d_{\text{loc}} \gamma[1] = 0\) so that \(1^2 = 0\).

(b) We define a pre-domain \(D_0\) dense in \(L^2(\mathcal{P})\) by

\[
D_0 = \left\{ \sum_{p=1}^m \lambda_p e^{iN(f_p)}; \ m \in \mathbb{N}^+, \lambda_p \in \mathbb{C}, f_p \in D(a) \cap L^1(v) \right\}.
\]

(c) We introduce the creation operator inspired from quantum mechanics (see [7–9,1,5,6] and [10] among others) defined as follows

\[
\epsilon^+_x(\omega) = 0 \quad v\text{-a.s.}
\]

so that

\[
\epsilon^+_x(\omega) = 0 \quad \mathbb{N}_\omega\text{-a.e.} \ x \quad \text{and} \quad \epsilon^+_x(\omega) = \omega + \epsilon_x \quad v\text{-a.e.} \ x.
\]

This map is measurable and the Laplace functional shows that for an \(\mathcal{A} \times \mathcal{X}\)-measurable \(H \geq 0,\)

\[
\mathbb{E}\int \epsilon^+ H \, dv = \mathbb{E}\int H \, dN.
\]

Let us remark also that by (5), for \(F \in L^2(\mathbb{P}_N \times \rho)\)

\[
\int \epsilon^+ F \, d(N \circ \rho) = \int F \, d(N \circ \rho) \quad \mathbb{P} \times \hat{\mathbb{P}}\text{-a.s.}
\]

(d) We defined a gradient \(\sharp\) for the up-structure on \(D_0\) by putting for \(F \in D_0\)

\[
F^\sharp = \int (\epsilon^+ F)^\flat \, d(N \circ \rho)
\]
this definition being justified by the fact that for \( \mathbb{P}\text{-a.e. } \omega \) the map \( y \mapsto F(\varepsilon^+_y(\omega)) - F(\omega) \) is in \( d \), \( \varepsilon^+ F \) belongs to \( L^\infty(\mathbb{P}) \otimes d \) algebraic tensor product, and \((\varepsilon^+ F - F)^\circ) \in L^2(\mathbb{P}_N \times \rho) \).

For \( F, G \in D_0 \) of the form
\[
F = \sum_p \lambda_p e^{i\tilde{N}(f_p)} = \Phi(\tilde{N}(f_1), \ldots, \tilde{N}(f_m)), \quad G = \sum_q \mu_q e^{i\tilde{N}(g_q)} = \Psi(\tilde{N}(g_1), \ldots, \tilde{N}(g_n))
\]
we compute using (2), (3) and (7) (in the spirit of Prop. 1 of [9] or Lemma 1.2 of [6])
\[
\mathbb{E}[F^\varepsilon G^\varepsilon] = \sum_{p,q} \lambda_p \mu_q e^{i\tilde{N}(f_p) - i\tilde{N}(g_q)} N(\gamma[f_p, g_q])
\]
and we have:

**Proposition 3.1.** If we put \( A_0[F] = \sum_p \lambda_p e^{i\tilde{N}(f_p)}(i\tilde{N}(a[f_p]) - \frac{1}{2}N(\gamma[f_p])) \) it comes
\[
\mathbb{E}[A_0[F]G] = -\frac{1}{2} \mathbb{E} \sum_{p,q} \Phi_p \overline{\Psi}_q N(\gamma[f_p, g_q]). \tag{10}
\]

In order to show that \( A_0[F] \) does not depend on the form of \( F \), by (10) it is enough to show that the expression \( \sum_{p,q} \Phi_p \overline{\Psi}_q N(\gamma[f_p, g_q]) \) depends only on \( F \) and \( G \). But this comes from (9) since \( F^\varepsilon \) and \( G^\varepsilon \) depend only on \( F \) and \( G \).

By this proposition, \( A_0 \) is symmetric on \( D_0 \), negative, and the argument of Friedrichs applies (cf. [3] p. 4). \( A_0 \) extends uniquely to a selfadjoint operator \((A, D(A))\) which defines a closed positive (hermitian) quadratic form \( E[F] = -\mathbb{E}[A[F]^2] \). By (10) contractions operate and (cf. [3]) \( E \) is a Dirichlet form which is local with carré du champ denoted \( \Gamma \) and the up-structure obtained \((\Omega, A, \mathbb{P}, D, \Gamma)\) satisfies
\[
\forall f \in d, \quad \tilde{N}(f) \in D \quad \text{and} \quad \Gamma[\tilde{N}(f)] = N(\gamma[f]). \tag{11}
\]
The operator \( \sharp \) extends to a gradient for \( \Gamma \) as a closed operator from \( L^2(\mathbb{P}) \) into \( L^2(\mathbb{P} \times \hat{\mathbb{P}}) \) with domain \( D \) which satisfies the chain rule and may be computed on functionals \( \Phi(\tilde{N}(f_1), \ldots, \tilde{N}(f_m)) \), \( \Phi \) Lipschitz and \( C^1 \) and their limits in \( D \) (as done in [4]).

Formula (8) for \( \sharp \) can be extended from \( D_0 \) to \( D \). Let us introduce the space \( D \) closure of \( D_0 \otimes d \) for the norm
\[
\| H \|_D = \left( \mathbb{E} \int \gamma[H(\omega, \cdot)](x) N(dx) \right)^{1/2} + \mathbb{E} \int [H(\omega, x)]\xi(x) N(dx)
\]
where \( \xi > 0 \) is a fixed function such that \( N(\xi) \in L^2(\mathbb{P}) \).

**Theorem 3.2.** The formula \( F^\varepsilon = \int (\varepsilon^+ F)^\circ d(N \otimes \rho) \) decomposes as follows
\[
F \in D \mapsto \varepsilon^+ F \in D \mapsto \int (\varepsilon^+ F)^\circ \in L^2_0(\mathbb{P}_N \times \rho) \overset{\text{d}(N \otimes \rho)}{\longrightarrow} F^\varepsilon \in L^2(\mathbb{P} \times \hat{\mathbb{P}})
\]
where each operator is continuous on the range of the preceding one, \( L^2_0(\mathbb{P}_N \times \rho) \) denoting the closed subspace of \( L^2(\mathbb{P}_N \times \rho) \) of \( \rho \)-centered elements, and we have
\[
\Gamma[F] = \mathbb{E}[F^\varepsilon^2] = \int \gamma[\varepsilon^+ F] dN. \tag{12}
\]

4. The lent particle method

Let us consider, for instance, a real process \( Y_t \) with independent increments and Lévy measure \( \sigma \) integrating \( x^2 \), \( Y_t \) being supposed centered without Gaussian part. We assume that \( \sigma \) has an l.s.c. density so that a local Dirichlet structure may be constructed on \( \mathbb{R} \setminus \{0\} \) with carré du champ \( \gamma[f] = x^2 f^2(x) \). If \( N \) is the random Poisson measure with intensity \( \delta \times \sigma \) we have \( \int_0^t h(s) \, dy_t = \int_1 [0,t]h(s) \tilde{N}(ds dx) \) and the choice done for \( \gamma \) gives \( \Gamma[\int_0^t h(s) \, dy_t] = \int_0^t h^2(x) \, dy_t \), for \( h \in L^2_0(\delta \times \sigma) \). In order to study the regularity of the random variable \( V = \int_0^T \varphi(Y_{s-}) \, dy_s \) where \( \varphi \) is Lipschitz and \( C^1 \), we have two ways:
(a) We may represent the gradient $\nabla Y$ as $Y^\sharp_t = B_t[Y,Y]$ where $B$ is a standard auxiliary independent Brownian motion. Then by the chain rule $V^\sharp_t = \int_0^t \phi'(Y_s) (Y_s - Y_s^\sharp) dY_s + \int_0^t \phi(Y_s) dB_s$ now, using $(Y_s^\sharp)^2 = (Y_s^\sharp)^2 - $, a classical but rather tedious stochastic computation yields

$$\Gamma[V] = \hat{E}[V^{\sharp 2}] = \sum_{\alpha \leq t} \Delta Y_\alpha^2 \left( \int_{[\alpha]} \phi'(Y_s) dY_s + \phi(Y_\alpha) \right)^2. \quad (13)$$

Since $V$ has real values the energy image density property holds, and $V$ has a density as soon as $\Gamma[V]$ is strictly positive a.s. what may be discussed using the relation (13).

(b) Another more direct way consists in applying the theorem. For this we define $b$ by choosing $\eta$ such that $\int_0^1 \eta(r) dr = 0$ and $\int_0^1 \eta^2(r) dr = 1$ and putting $f^\sharp = xf'(x)\eta(r)$.

1°. First step. We add a particle $(\alpha,x)$ i.e. a jump to $Y$ at time $\alpha$ with size $x$ what gives $\varepsilon V - V = \varphi(Y_\alpha) x + \int_\alpha^t \varphi'(Y_s) x dY_s$.

2°. $V^\sharp = 0$ since $V$ does not depend on $x$, and $(\varepsilon V)^\sharp = (\varphi(Y_\alpha)x + \int_\alpha^t \varphi'(Y_s) x dY_s)\eta(r)$ because $x^\sharp = x\eta(r)$.

3°. We compute $\gamma[\varepsilon V] = \int (\varepsilon V)^\sharp)^2 dr = (\varphi(Y_\alpha)x + \int_\alpha^t \varphi'(Y_s) x dY_s)^2$.

4°. We take back the particle we gave, because in order to compute $\int \gamma[\varepsilon V] dN$ the integral in $N$ confuses $\varepsilon$ and $\omega$. That gives $\int \gamma[\varepsilon V] dN = \int (\varphi(Y_\alpha) + \int_\alpha^t \varphi'(Y_s) dY_s)^2 x^2 N(\omega)$ and (13).

We remark that both operators $F \mapsto \varepsilon F$, $F \mapsto (\varepsilon F)^\sharp$ are non-local, but instead $F \mapsto \int (\varepsilon F)^\sharp d(N \otimes \rho)$ and $F \mapsto \int \gamma[\varepsilon F] dN$ are local: taking back the lent particle gives the locality.

References