Abstract

In this Note, we determine all the totally positive integers of \( \mathbb{Q}(\sqrt{5}) \) which cannot be represented as sums of distinct integral squares. To cite this article: P.-S. Park, C. R. Acad. Sci. Paris, Ser. I 346 (2008). © 2008 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

1. Introduction

After Lagrange’s Four Squares Theorem many mathematicians studied sums of squares in various algebraic number fields. Let \( F \) be an algebraic number field of degree \( n = r_1 + 2r_2 \) with \( r_1 \) real embeddings and \( r_2 \) pairs of complex embeddings. If all embeddings of \( F \) are real, that is, \( n = r_1 \), then we call \( F \) totally real. If an element \( \alpha \in F \) satisfies \( \sigma(\alpha) > 0 \) for all real embeddings \( \sigma \), we call \( \alpha \) totally positive. In 1902 Hilbert asked whether every totally positive integer of \( F \) is a sum of four squares in \( F \). Götzky answered by the surprising theorem [1]:

**Theorem 1.1.** The field \( \mathbb{Q}(\sqrt{5}) \) is the only real quadratic field in which every totally positive integer can be represented as a sum of four integral squares.

This was improved by Maass to reduce the number of squares [3]:

**Theorem 1.2.** In \( \mathbb{Q}(\sqrt{5}) \) every totally positive integer can be represented as a sum of three integral squares.

Furthermore, Siegel proved the remarkable theorem [4]:

**Résumé**

Sommes de carrés distinctes dans l’anneau d’entiers de \( \mathbb{Q}(\sqrt{5}) \). Nous déterminons tous les entiers totalement positifs qui ne peuvent pas être représentés comme des sommes de carrés distincts d’entiers dans \( \mathbb{Q}(\sqrt{5}) \). Pour citer cet article : P.-S. Park, C. R. Acad. Sci. Paris, Ser. I 346 (2008). © 2008 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.
Theorem 1.3. If in a totally real field \( F \), every totally positive integer can be represented as a sum of integral squares, then \( F \) is either \( \mathbb{Q} \) or \( \mathbb{Q}(\sqrt{5}) \).

After a few years Sprague [5] showed the following theorem:

Theorem 1.4. Every positive integer in \( \mathbb{Q} \) larger than 128 can be represented as a sum of distinct integral squares.

There are 31 positive integers which cannot be represented as sums of distinct squares: 2, 3, 6, 7, 8, 11, 12, 15, 18, 19, 22, 23, 24, 27, 28, 31, 32, 33, 43, 44, 47, 48, 60, 67, 72, 76, 92, 96, 108, 112 and 128.

Now let us consider the field \( \mathbb{Q}(\sqrt{5}) \), the other totally real field in which sums of squares represent all totally positive integers.

2. Result

The ring of integers of the number field \( \mathbb{Q}(\sqrt{5}) \) is generated by 1 and the fundamental unit \( \epsilon = \frac{1+\sqrt{5}}{2} \) and also by 1 and \( \epsilon^2 = \frac{3+\sqrt{5}}{2} \). The totally positive integers in \( \mathbb{Q}(\sqrt{5}) \) can be characterized as in the following lemma due to Kim [2]. We give a simpler proof here.

Lemma 2.1. Every totally positive integer in \( \mathbb{Q}(\sqrt{5}) \) is of the form \( \epsilon^{2n}(a + be^2) \) for some \( n \in \mathbb{Z} \) and nonnegative integers \( a, b \).

Proof. Let \( S \) be the set of totally positive integers of the form stated above. Suppose that there exist totally positive integers not belonging to \( S \). Choose \( a = a + be^2 \) with minimal trace among those. Then \( a \) and \( b \) should have opposite signs. If \( a < 0 < b \), then \( (a + be^2)\epsilon^{-2} = b + ae^{-2} \) is of trace \( 2b + 3a \), which is smaller than \( \text{Tr}(a) = 2a + 3b \). Hence \( (a + be^2)\epsilon^{-2} \in S \) by the minimality condition. This means \( a \in S \), which is a contradiction.

Now assume \( a > 0 > b \). Consider \( a\epsilon^2 = -b + (a + 3b)\epsilon^2 \). If \( a + 3b \geq 0 \), we are finished. If \( a + 3b < 0 \), then \( \text{Tr}(a\epsilon^2) = 3a + 7b < 2a + 3b \). By the minimality condition, \( (a + be^2)\epsilon^2 \in S \). This is also a contradiction. Hence the lemma is proved. \( \square \)

Lemma 2.2. Any nonnegative rational integer except 2, 3, 7, 8, 11, 12, 23, 27, 28, 32, 48 can be represented as a sum of distinct integral squares in \( \mathbb{Q}(\sqrt{5}) \).

Proof. Let \( T \) be the set of rational integers which can be represented as a sum of distinct squares in \( \mathbb{Z} \). Then \( T = \{0, 1, 4, 5, 9, 10, 13, 14, 16, 17, 20, \ldots \} \) according to Sprague’s result. Since \( (1 - 2\epsilon)^2 = 5 \), \( a + 5b \) with \( a, b \in T \) can be represented as sums of distinct integral squares. For example, \( 128 = (3^2 + 7^2) + 5(1^2 + 2^2 + 3^2) \). The nonnegative integers not represented in this form are 2, 3, 7, 8, 11, 12, 23, 27, 28, 32 and 48. \( \square \)

Theorem 2.3. Let \( \alpha \) be a totally positive integer in \( \mathbb{Q}(\sqrt{5}) \). Then, \( \alpha \) cannot be represented as a sum of distinct squares if and only if the norm of \( \alpha \) is one of 4, 11, 19, 44 and 59.

Proof. We may consider the number \( a + be^2 \) with \( 0 \leq a \leq b \) since \( \epsilon \) is a unit and \( b + ae^2 = (a + be^2)\epsilon^2 \). If \( 49 \leq a \leq b \), we are finished by Lemma 2.2. Assume \( 0 \leq a \leq 48 \) and \( 51 \leq b \). We have \( a + be^2 - (1 + e)^2 - (1 - e)^2 = (a - 2) + (b - 2)\epsilon^2 \). Since \( a \) or \( a - 2 \) can be represented as sums of distinct squares by Lemma 2.2, this case is solved.

Thus we now investigate the case \( 0 \leq a \leq b \leq 50 \). All except \( 2\epsilon^2, 1 + 2\epsilon^2, 1 + 3\epsilon^2, 2 + 4\epsilon^2 \) and \( 2 + 5\epsilon^2 \) can be solved by using one of the following relations:

\[
\begin{align*}
(1 + \epsilon)^2 &= -1 + 3\epsilon^2; \quad (1 - \epsilon)^2 = 3 - \epsilon^2; \quad (1 + 2\epsilon)^2 = -3 + 8\epsilon^2; \\
(1 + \epsilon)^2 + (2 - \epsilon)^2 &= 7; \quad (1 - \epsilon)^2 + (1 + 2\epsilon)^2 = 7\epsilon^2; \\
(1 + \epsilon)^2 + (1 - \epsilon)^2 &= 2 + 2\epsilon^2; \quad (1 + \epsilon)^2 + (3 + \epsilon)^2 = 2 + 10\epsilon^2.
\end{align*}
\]

The exceptions are of norm 4, 11, 19, 44 and 59, respectively, and they cannot be represented as sums of distinct squares. We only prove this for \( 2 + 5\epsilon^2 \) since the other four exceptions can be proved in a similar manner.
Suppose that $2 + 5\varepsilon^2$ can be represented as a sum of $n$ distinct squares. That is,

$$2 + 5\varepsilon^2 = \sum_{i=1}^{n} (a_i + b_i\varepsilon)^2$$

for some $a_i, b_i \in \mathbb{Z}$. Then an easy calculation shows that

$$2 + 5 = \sum_{i=1}^{n} (a_i^2 + b_i^2).$$

All the possible pairs $(a_i, b_i)$ are $(2, 1), (2, -1), (2, 0), (1, 2), (1, -2), (1, 1), (1, -1), (1, 0), (0, 2)$ and $(0, 1)$. The squares corresponding to these pairs are $5\varepsilon^2, 8 - 3\varepsilon^2, 4, -3 + 8\varepsilon^2, 5, -1 + 3\varepsilon^2, 3 - \varepsilon^2, 1, 4\varepsilon^2$ and $\varepsilon^2$, respectively. One can easily check, by comparing traces, that sums of all or parts of these squares can never be equal to $2 + 5\varepsilon^2$.

Furthermore, the exceptions are the only totally positive integers having those norms up to multiplication by $\varepsilon^{2n}$. Hence the theorem is proved. □

Note that in the theorem above, we do not claim that three distinct squares suffice. The situation is similar to that of $\mathbb{Q}$ where four distinct squares do not suffice (for instance, $132 = 9^2 + 5^2 + 4^2 + 3^2 + 1^2$ needs five squares).

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References