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Number Theory

Sums of distinct integral squares in $\mathbb{Q}(\sqrt{5})$

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Abstract

In this Note, we determine all the totally positive integers of $\mathbb{Q}(\sqrt{5})$ which cannot be represented as sums of distinct integral squares. ***To cite this article: P.-S. Park, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Sommes de carrés distinctes dans l'anneau d'entiers de $\mathbb{Q}(\sqrt{5})$. Nous déterminons tous les entiers totalement positifs qui ne peuvent pas être représentés comme des sommes de carrés distincts d'entiers dans $\mathbb{Q}(\sqrt{5})$. ***Pour citer cet article : P.-S. Park, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

1. Introduction

After Lagrange's Four Squares Theorem many mathematicians studied sums of squares in various algebraic number fields. Let F be an algebraic number field of degree $n = r_1 + 2r_2$ with r_1 real embeddings and r_2 pairs of complex embeddings. If all embeddings of F are real, that is, $n = r_1$, then we call F *totally real*. If an element $\alpha \in F$ satisfies $\sigma(\alpha) > 0$ for all real embeddings σ, we call α *totally positive*. In 1902 Hilbert asked whether every totally positive integer of F is a sum of four squares in F. Götzky answered by the surprising theorem [1]:

Theorem 1.1. *The field $\mathbb{Q}(\sqrt{5})$ is the only real quadratic field in which every totally positive integer can be represented as a sum of four integral squares.*

This was improved by Maass to reduce the number of squares [3]:

Theorem 1.2. *In $\mathbb{Q}(\sqrt{5})$ every totally positive integer can be represented as a sum of three integral squares.*

Furthermore, Siegel proved the remarkable theorem [4]:

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Theorem 1.3. *If in a totally real field F, every totally positive integer can be represented as a sum of integral squares, then F is either $\mathbb{Q}$ or $\mathbb{Q}(\sqrt{5})$.*

After a few years Sprague [5] showed the following theorem:

Theorem 1.4. *Every positive integer in $\mathbb{Q}$ larger than* 128 *can be represented as a sum of distinct integral squares.*

There are 31 positive integers which cannot be represented as sums of distinct squares: 2, 3, 6, 7, 8, 11, 12, 15, 18, 19, 22, 23, 24, 27, 28, 31, 32, 33, 43, 44, 47, 48, 60, 67, 72, 76, 92, 96, 108, 112 and 128.

Now let us consider the field $\mathbb{Q}(\sqrt{5})$, the other totally real field in which sums of squares represent all totally positive integers.

2. Result

The ring of integers of the number field $\mathbb{Q}(\sqrt{5})$ is generated by 1 and the fundamental unit $\epsilon = \frac{1+\sqrt{5}}{2}$ and also by 1 and $\epsilon^2 = \frac{3+\sqrt{5}}{2}$. The totally positive integers in $\mathbb{Q}(\sqrt{5})$ can be characterized as in the following lemma due to Kim [2]. We give a simpler proof here.

Lemma 2.1. *Every totally positive integer in $\mathbb{Q}(\sqrt{5})$ is of the form $\epsilon^{2n}(a + b\epsilon^2)$ for some $n \in \mathbb{Z}$ and nonnegative integers a, b.*

Proof. Let S be the set of totally positive integers of the form stated above. Suppose that there exist totally positive integers not belonging to S. Choose $\alpha = a + b\epsilon^2$ with minimal trace among those. Then a and b should have opposite signs. If $a < 0 < b$, then $(a + b\epsilon^2)\epsilon^{-2} = b + a\epsilon^{-2}$ is of trace $2b + 3a$, which is smaller than $\mathrm{Tr}(\alpha) = 2a + 3b$. Hence $(a + b\epsilon^2)\epsilon^{-2} \in S$ by the minimality condition. This means $\alpha \in S$, which is a contradiction.

Now assume $a > 0 > b$. Consider $\alpha\epsilon^2 = -b + (a + 3b)\epsilon^2$. If $a + 3b \geqslant 0$, we are finished. If $a + 3b < 0$, then $\mathrm{Tr}(\alpha\epsilon^2) = 3a + 7b < 2a + 3b$. By the minimality condition, $(a + b\epsilon^2)\epsilon^2 \in S$. This is also a contradiction. Hence the lemma is proved. □

Lemma 2.2. *Any nonnegative rational integer except* 2, 3, 7, 8, 11, 12, 23, 27, 28, 32, 48 *can be represented as a sum of distinct integral squares in $\mathbb{Q}(\sqrt{5})$.*

Proof. Let T be the set of rational integers which can be represented as a sum of distinct squares in $\mathbb{Z}$. Then $T = \{0, 1, 4, 5, 9, 10, 13, 14, 16, 17, 20, \ldots\}$ according to Sprague's result. Since $(1 - 2\epsilon)^2 = 5$, $a + 5b$ with $a, b \in T$ can be represented as sums of distinct integral squares. For example, $128 = (3^2 + 7^2) + 5(1^2 + 2^2 + 3^2)$. The nonnegative integers not represented in this form are 2, 3, 7, 8, 11, 12, 23, 27, 28, 32 and 48. □

Theorem 2.3. *Let α be a totally positive integer in $\mathbb{Q}(\sqrt{5})$. Then, α cannot be represented as a sum of distinct squares if and only if the norm of α is one of* 4, 11, 19, 44 *and* 59.

Proof. We may consider the number $a + b\epsilon^2$ with $0 \leqslant a \leqslant b$ since ϵ is a unit and $b + a\epsilon^2 = (\overline{a + b\epsilon^2})\epsilon^2$. If $49 \leqslant a \leqslant b$, we are finished by Lemma 2.2. Assume $0 \leqslant a \leqslant 48$ and $51 \leqslant b$. We have $a + b\epsilon^2 - (1 + \epsilon)^2 - (1 - \epsilon)^2 = (a - 2) + (b - 2)\epsilon^2$. Since a or $a - 2$ can be represented as sums of distinct squares by Lemma 2.2, this case is solved.

Thus we now investigate the case $0 \leqslant a \leqslant b \leqslant 50$. All except $2\epsilon^2$, $1 + 2\epsilon^2$, $1 + 3\epsilon^2$, $2 + 4\epsilon^2$ and $2 + 5\epsilon^2$ can be solved by using one of the following relations:

$$(1 + \epsilon)^2 = -1 + 3\epsilon^2; \quad (1 - \epsilon)^2 = 3 - \epsilon^2; \quad (1 + 2\epsilon)^2 = -3 + 8\epsilon^2;$$
$$(1 + \epsilon)^2 + (2 - \epsilon)^2 = 7; \quad (1 - \epsilon)^2 + (1 + 2\epsilon)^2 = 7\epsilon^2;$$
$$(1 + \epsilon)^2 + (1 - \epsilon)^2 = 2 + 2\epsilon^2; \quad (1 + \epsilon)^2 + (3 + \epsilon)^2 = 2 + 10\epsilon^2.$$

The exceptions are of norm 4, 11, 19, 44 and 59, respectively, and they cannot be represented as sums of distinct squares. We only prove this for $2 + 5\epsilon^2$ since the other four exceptions can be proved in a similar manner.

Suppose that $2+5\epsilon^2$ can be represented as a sum of n distinct squares. That is,

$$2+5\epsilon^2=\sum_{i=1}^{n}(a_i+b_i\epsilon)^2$$

for some $a_i, b_i \in \mathbb{Z}$. Then an easy calculation shows that

$$2+5=\sum_{i=1}^{n}(a_i^2+b_i^2).$$

All the possible pairs (a_i, b_i) are $(2,1)$, $(2,-1)$, $(2,0)$, $(1,2)$, $(1,-2)$, $(1,1)$, $(1,-1)$, $(1,0)$, $(0,2)$ and $(0,1)$. The squares corresponding to these pairs are $5\epsilon^2$, $8-3\epsilon^2$, 4, $-3+8\epsilon^2$, 5, $-1+3\epsilon^2$, $3-\epsilon^2$, 1, $4\epsilon^2$ and ϵ^2, respectively. One can easily check, by comparing traces, that sums of all or parts of these squares can never be equal to $2+5\epsilon^2$.

Furthermore, the exceptions are the only totally positive integers having those norms up to multiplication by ϵ^{2n}. Hence the theorem is proved. □

Note that in the theorem above, we do not claim that *three* distinct squares suffice. The situation is similar to that of $\mathbb{Q}$ where four distinct squares do not suffice (for instance, $132=9^2+5^2+4^2+3^2+1^2$ needs five squares).

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