Algebra

Extensions with Galois group $2^+S_4 \ast D_8$ in characteristic 3

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Abstract

For $K$ a field of characteristic 3 we give explicitly the whole family of Galois extensions of $K$ with Galois group $2^+S_4 \ast D_8$ and determine the discriminant of such an extension. In the case when $K$ is the field of fractions of a formal power series ring in 3 variables, this result is interesting in the context of Abhyankar’s Normal Crossings Local Conjecture. To cite this article: T. Crespo, Z. Hajto, C. R. Acad. Sci. Paris, Ser. I 346 (2008).

Résumé


1. Introduction

In this Note we give explicitly the whole family of Galois extensions of a field $K$ of characteristic 3 with Galois group $2^+S_4 \ast D_8$ and determine the discriminant of such an extension. This result improves the one obtained in [3] by dropping the condition assumed there that the fields considered contain the field $\mathbb{F}_9$ of nine elements. In the case when $K$ is the field of fractions of the formal power series ring in 3 variables over a field $k$ of characteristic 3, the explicit determination of its $2^+S_4 \ast D_8$-coverings and their discriminant is interesting in the context of Abhyankar’s Normal Crossings Local Conjecture (see [2,4] as well as the Introduction in [3]).

2. Preliminaries

We denote by $2^+S_n$ the double cover of the symmetric group $S_n$ in which transpositions lift to involutions and products of two disjoint transpositions lift to elements of order 4 and by $D_8$ the dihedral group of order 8, which is a

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double cover of the Klein group $V_4$. We denote by $2^+ S_4 \ast D_8$ the central product of the groups $2^+ S_4$ and $D_8$. Let $K$ be a field of characteristic different from 2 and let $L/K$ be a Galois extension with Galois group the group $2^+ S_4 \ast D_8$. Then if $L$ is the field fixed by the center of $2^+ S_4 \ast D_8$, we have $\text{Gal}(L/K) \cong S_6 \times V_4$ and for $L_1$, $L_2$ the fixed subfields of $L$ by $V_4$ and $S_6$, respectively, we have $\text{Gal}(L_1/K) \cong S_6$ and $\text{Gal}(L_2/K) \cong V_4$. Therefore we obtain the whole family of Galois extensions with Galois group $2^+ S_4 \ast D_8$ of a field $K$ by constructing the whole family of $2^+ S_4 \ast D_8$-extensions containing a given arbitrary $S_4$-extension of the field $K$. Let us now be given a polynomial $f(X) \in K[X]$ of degree 4 with Galois group $S_4$ and splitting field $L_1$ over $K$. We want to determine when $L_1$ is embeddable in a Galois extension of $K$ with Galois group $2^+ S_4 \ast D_8$. This fact is equivalent to the existence of a Galois extension $L_2/K$ with Galois group $V_4$, disjoint from $L_1$, and such that, if $L$ is the compositum of $L_1$ and $L_2$, the Galois embedding problem

$$2^+ S_4 \ast D_8 \to S_4 \times V_4 \cong \text{Gal}(L/K)$$

is solvable. We recall that a solution to this embedding problem is a quadratic extension $\tilde{L}$ of the field $L$, which is a Galois extension of $K$ with Galois group $2^+ S_4 \ast D_8$ and such that the restriction epimorphism between the Galois groups $\text{Gal}(\tilde{L}/K) \to \text{Gal}(L/K)$ agrees with the given epimorphism $2^+ S_4 \ast D_8 \to S_4 \times V_4$. If $\tilde{L} = L(\sqrt{y})$ is a solution, then the general solution is $L(\sqrt{y^r})$, $r \in K^\times$. Given a Galois extension $L_1/K$ with Galois group $S_4$, in order to obtain all $2^+ S_4 \ast D_8$-extensions of $K$ containing $L_1$, we have to determine all $V_4$-extensions $L_2$ of $K$, disjoint from $L_1$, and such that the embedding problem (1) is solvable.

Let $E = K[X]/(f(X))$, for $f(X)$ the polynomial of degree 4 realizing $L_1$ and let $d$ be the discriminant of the polynomial $f(X)$. Let $L_2 = K(\sqrt{\alpha}, \sqrt{\beta})$. The obstruction to the solvability of the embedding problem (1) is equal to $w(Q_E) \times d/mol(-1)$, where $Q_E$ denotes the trace form of the extension $E/K$ and $(\cdot, \cdot)$ a Hilbert symbol (see [3]).

From now on, we assume that $K$ is a field of characteristic 3. We write $f(X) = X^4 + s_2 X^2 - s_3 X + s_4$. By computation of the trace form $Q_E$, we obtain that the solvability of the embedding problem (1) is equivalent to

$$(-ds_2, -(s_2^2 - s_3) ds_2) = (a, b).$$

(2)

3. Main results

**Theorem 3.1.** Let $K$ be a field of characteristic 3, $f(X) = X^4 + s_2 X^2 - s_3 X + s_4 \in K[X]$, with Galois group $S_4$ and $L_1$ the splitting field of $f(X)$ over $K$. Let $d = s_2^2 + s_2^2 s_3^2 + s_2^2 s_4 - s_3^2 s_2^2$ the discriminant of the polynomial $f(X)$. The family of elements $a, b$ in $K$ such that $(a, b) = (-ds_2, -ms_2)$, where $m := s_2^2 - s_4$, can be given in terms of an arbitrary invertible matrix $P = (p_{ij})_{1 \leq i, j, k \leq 3} \in GL(3, K)$ as $a = -dA$, $b = -s_2mF$, where

$$A = s_2 P_{11}^2 + mp_{21}^2 + dms_2 P_{31}^2,$n

$$F = dm P_{13}^2 + ds_2 P_{23}^2 + P_{33}^2,$n

with $P_{ij} = \begin{vmatrix} p_{ij} & p_{ij} \\ p_{ji} & p_{jj} \end{vmatrix}.$n

Let $L_2 = K(\sqrt{\alpha}, \sqrt{\beta})$ and assume that $L_2/K$ has Galois group $V_4$ and $L_1 \cap L_2 = K$ (i.e. that the elements $a, b, ab, da, db, dab$ are not squares in $K$). Let $L = L_1 \cdot L_2$. For $x$ a root of the polynomial $f(X)$, take $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$, with

$$a_0 = -s_2 s_2,$n

$$a_1 = -dm(ns_2 p_{11} P_{23} + p_{21} P_{33} + mp_{21} P_{13} + ds_2 P_{31} P_{23}) + m \sqrt{\alpha} (d P_{13} - n P_{33}),$$n

$$a_2 = ds_2 (p_{11} P_{33} + s_2^2 p_{11} P_{23} + s_4^2 s_3 p_{21} P_{23} + ms_2 s_3 p_{21} P_{13} - dmp_{31} P_{13} + s_2 \sqrt{\alpha} (s_2 s_3 P_{33} - d P_{23}),$$n

$$a_3 = -dms_2 (s_2 p_{11} P_{23} + mp_{21} P_{13}) - ms_2 \sqrt{\alpha} P_{33},$$n

where $n = s_2^2 + s_4$. Then $L(\sqrt{y^r})$, $r \in K^\times$, is the general solution to the embedding problem

$$2^+ S_4 \ast D_8 \to S_4 \times V_4 \cong \text{Gal}(L/K).$$

**Proof.** By [5], 3.2, the equality of Hilbert symbols (2) is equivalent to the $K$-equivalence of quadratic forms

$$\langle -ds_2, -ms_2, -dm \rangle \sim \langle a, b, -ab \rangle.$$

(3)
The family of quadratic forms $K$-equivalent to $R := (-d s_2, -m s_2, -d m)$ is given by $P^T R P$, for $P$ running over $GL(3, K)$. By diagonalizing $P^T R P$, we obtain $(-d A, -s_2 m F, -d A s_2 m F)$, with $A$ and $F$ as in the statement. Let $a = -d A, b = -s_2 m F$. Now, we have $(a, b) = 1 \in H^2(G, K[\sqrt{a}], \langle \pm 1 \rangle)$ and, as $a \notin K^2$ and $L_1 \cap K(\sqrt{a}) = K$, the extension $L_1(\sqrt{a}) K(\sqrt{a})$ has Galois group $S_4$ and the Galois embedding problem $2^+ S_4 \to S_4 \simeq \text{Gal}(L_1(\sqrt{a}) K(\sqrt{a}))$ is soluble. Then, by Abhyankar’s Embedding Criterion (see [1,3]), $L_1(\sqrt{a})$ is the splitting field of a polynomial of the form $Y^4 + c_3 Y + c_4 \in K(\sqrt{a})[Y]$, so there exists elements $a_0, a_1, a_2, a_3 \in K(\sqrt{a})$ such that the irreducible polynomial over $K(\sqrt{a})$ of the element $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ has such a form. By computation, this is equivalent to the conditions $a_0 = -a_2 a_3$ and $Q(a_1, a_2, a_3) := s_2 a_1^2 + (s_2^2 - s_4) a_2^2 + s_2^2 a_3^2 + (s_2^2 + s_4) a_1 a_3 + 2 s_2 s_3 a_2 a_3 = 0$. Moreover, by Abhyankar’s Polynomial Theorem (see [1,3]), the splitting field of the polynomial $\text{Irr}(K, (\sqrt{a}))$, that is the field $L_1(\sqrt{a}) K(\sqrt{a})$, is a solution to the Galois embedding problem $2^+ S_4 \to S_4 \simeq \text{Gal}(L_1(\sqrt{a}) K(\sqrt{a}))$. Our aim now is to compute explicitly such elements $a_i$. Diagonalizing $Q$, we obtain $(s_2, m, s_2 m d) \sim (A, s_2 m A F, s_2 m F d)$ and the basis change matrix can be written down explicitly in terms of the matrix $P$. Now the vector $(0, d, \sqrt{a}) \in K(\sqrt{a})^3$ annihilates the quadratic form $(A, s_2 m A F, s_2 m F d)$ and from it we obtain the values for $a_1, a_2, a_3 \in K(\sqrt{a})$ such that $Q(a_1, a_2, a_3) = 0$.

We want to see now that $L(\sqrt{a}) K(\sqrt{a})$ is a Galois extension with Galois group $2^+ S_4 \rtimes D_8$. By the assumption $L_1 \cap L_2 = K$, we have $\text{Gal}(L_1(\sqrt{a}) K(\sqrt{a})) \simeq 2^+ S_4$. We consider now the behaviour of $y$ under the action of $\text{Gal}(L_2/K)$. Let $s, t$ be the non-trivial elements of $\text{Gal}(L_2/K)$ fixing respectively $\sqrt{ab}$, $\sqrt{b}$, $\sqrt{a}$. By computation we obtain $y^2 = d^2 t^2 b$, where $h = m s_2 s_3^3 \cdot (p_{21} + s_2^2 s_3 s_1^2) + (m n p_{21} - p_{21}) x + s_2 s_3 s_1 + p_{21}$. Now $y \in K(\sqrt{a})(x)$, so $y^2 = y$ and $y^3 = y^4$, so $L(\sqrt{a})$ is Galois over $K$. Now we have $(d h \sqrt{b})^3 = d h \sqrt{b}$ and $(d h \sqrt{b})^3 = -d h \sqrt{b}$, so $\text{Gal}(L(\sqrt{a})) L_1(\sqrt{a b})$ cyclic, hence $\text{Gal}(L(\sqrt{a})) K(\sqrt{a}) \simeq 2^+ S_4 \rtimes D_8$. $\square$

**Proposition 3.2.** Let the fields $K$ and $L$ and the elements $s_2$, $s_3$, $s_4$, $d$, $a$, $b$, $m$, $p_{ij}$ and $y$ be as in Theorem 3.1. We have

$$\text{disc}(L(\sqrt{a}) K) = d^{44} a^{96} b^{120} D^{12}$$

where

$$D = s_4 p_{11}^4 - s_2 s_3 p_{11}^2 p_{21} + m s_2 s_1^2 p_{11}^2 - m s_3 p_{11} p_{21} + (m^2 - s_2 s_3^2) p_{21}^2
+ dp_{31} \left(p_{11}^2 + m s_2 s_1^2 p_{31} + s_3 p_{21}^2 + m^2 s_2 s_3 p_{21}^2\right) - d^2 p_{31}^2 (s_2 s_3 p_{21} + m n p_{11}) + d^3 p_{31}^4.$$

**Proof.** We have $\text{disc}(L(\sqrt{a}) K) = \text{disc}(L/K)^2 \cdot N_{L/K}(y)$ and $\text{disc}(L/K) = (d a b)^{48}$. Now

$$N_{L/K}(y) = \left(N_{L_1(\sqrt{a}) K}(y)^2\right)^2$$

and

$$N_{L_1(\sqrt{a}) K}(y) = N_{L_1(K)}(N_{L_1(\sqrt{a}) L_1}(y)) = N_{L_1(K)}(d^2 h^2 b) = d^{48} b^{24} N_{L_1(K)}(h)^2,$$

for $h$ as in the proof of Theorem 3.1. By computation, we obtain $N_{L_1(K)}(h) = D^6$, for $D$ as in the statement. $\square$

**4. Example**

Let $K = k((Z_1, Z_2, Z_3))$ be the field of fractions of the formal power series ring in 3 variables over a field $k$ of characteristic 3. We consider the family of polynomials $f_l(X) = X^4 + Z_1 X^2 + Z_2 X + Z_3 \in K[X]$, where $l$ is a positive integer number, i.e. we are taking $s_2 = Z_1$, $s_3 = -Z_2$, $s_4 = Z_3$. We can check that the polynomial $f_l$ has Galois group $S_4$ over $K$, for all $l \in \mathbb{N}_3$, and let $L_1$ be the splitting field of $f$ over $K$. We consider the extension $L_2/K$ generated by the elements $\sqrt{-d s_2}, \sqrt{-m s_2}, \sqrt{-d m}$. We can check that the elements $-d s_2, -m s_2, -d m, -s_2, -m s_2, -m$ are not squares in $K$ and so, $L_2/K$ has Galois group $V_4$ and is disjoint with $L_1/K$. Let $L = L_1 \cdot L_2$. Let $y$ be the element given by Theorem 3.1 for the matrix $P$, such that $p_{12} = p_{23} = p_{31} = 1$ and the other entries are equal to zero. Then we have $\text{Gal}(L(\sqrt{a}) K) \simeq 2^+ S_4 \rtimes D_8$, with $L(\sqrt{a}) L_1(\sqrt{-d s_2})$ cyclic. By applying Proposition 3.2, we see that the discriminant locus remains unchanged when going from $L$ to $L(\sqrt{a})$. 


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