Available online at www.sciencedirect.com
ScienceDirect
C. R. Acad. Sci. Paris, Ser. I 346 (2008) 603–606
COMPTES RENDUS MATHEMATIQUE
http://france.elsevier.com/direct/CRASS1/

Number Theory/Geometry

Congruence obstructions to pseudomodularity of Fricke groups

David Fithian

Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104-6395, USA

Received 16 August 2007; accepted after revision 14 April 2008
Available online 15 May 2008
Presented by Jean-Pierre Serre

Abstract

A pseudomodular group is a finite coarea non-arithmetic Fuchsian group whose set of cusps is $\mathbb{P}^1(\mathbb{Q})$. Long and Reid constructed finitely many of these by considering Fuchsian groups uniformizing one-cusped tori, i.e., Fricke groups. We show that a zonal (i.e., having a cusp at infinity) Fricke group with rational cusps is pseudomodular if and only if its set of finite cusps is dense in the finite adeles of $\mathbb{Q}$, and that there are infinitely many Fricke groups with rational cusps that are neither pseudomodular nor arithmetic. ***To cite this article: D. Fithian, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Obstacles à la pseudo-modularité des groupes de Fricke données par des conditions de congruence. Un groupe pseudo-modulaire est un groupe fuchsien, non-arithmétique et de coaire finie dont l'ensemble des pointes est $\mathbb{P}^1(\mathbb{Q})$. Long et Reid en ont construit un nombre fini en considérant les groupes fuchsiens qui uniformisent les tores à un trou, appelés groupes de Fricke. Nous démontrons ici qu'un groupe de Fricke, dont les pointes sont les nombres rationnels et l'infini, est pseudo-modulaire si et seulement si l'ensemble de ses pointes finies est dense dans le groupe des adèles finies de $\mathbb{Q}$. Nous en déduisons, l'existence d'une infinité de groupes de Fricke à pointes rationnelles, qui ne sont ni pseudo-modulaires ni arithmétiques. ***Pour citer cet article : D. Fithian, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

1. Introduction

A *cusp* of a Fuchsian group $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ is an $x \in \mathbb{P}^1(\mathbb{R})$ that is the unique fixed point of an element of Γ (see [6] Ch. 1). The modular group $\mathrm{PSL}_2(\mathbb{Z})$ is a finite coarea Fuchsian group whose set of cusps coincides with $\mathbb{P}^1(\mathbb{Q})$. In [5], Long and Reid show that there exist finite coarea Fuchsian subgroups of $\mathrm{PSL}_2(\mathbb{Q})$ that are *not* commensurable with $\mathrm{PSL}_2(\mathbb{Z})$ (i.e., not arithmetic) and whose cusp set equals $\mathbb{P}^1(\mathbb{Q})$. They call such groups *pseudomodular*.

Long and Reid studied a particular family $\Delta(u^2, 2t)$ of Fricke groups as candidates for pseudomodularity. *Fricke groups* are those Fuchsian groups that uniformize one-cusped hyperbolic tori; see [1]. As in [5], for rationals u^2 and t with $0 < u^2 < t - 1$, the group $\Delta(u^2, 2t)$ is the subgroup of $\mathrm{PSL}_2(\mathbb{R})$ freely generated by the hyperbolic elements

E-mail address: fithian@math.upenn.edu.

1631-073X/$ – see front matter
doi:10.1016/j.crma.2008.04.005

$$g_1 = \frac{1}{\sqrt{-1+t-u^2}}\begin{pmatrix} t-1 & u^2 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad g_2 = \frac{1}{u\sqrt{-1+t-u^2}}\begin{pmatrix} u^2 & u^2 \\ 1 & t-u^2 \end{pmatrix}.$$

Each such $\Delta(u^2, 2t)$ is a zonal Fricke group with exactly one orbit of cusps, which is contained in $\mathbb{P}^1(\mathbb{Q})$. (We call a Fuchsian group *zonal* if ∞ is among its cusps.) Moreover, every Fricke group whose cusps lie in $\mathbb{P}^1(\mathbb{Q})$ is conjugate in $\mathrm{PGL}_2(\mathbb{Q})$ to some $\Delta(u^2, 2t)$. This follows from a straightforward application of results in §1 of [3] to the traces of g_1, g_2 and g_1g_2. Thus the family of groups $\Delta(u^2, 2t)$ represents all conjugacy classes of Fricke groups having only rational cusps.

Among Long and Reid's stated open problems in [5] is the determination of the values $(u^2, 2t) \in \mathbb{Q} \times \mathbb{Q}$ for which $\Delta(u^2, 2t)$ is pseudomodular. Recall that if $\widehat{\mathbb{Z}}$ is the profinite completion of $\mathbb{Z}$, then $\mathbb{A}_{\mathbb{Q},f} = \mathbb{Q} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ is an additive topological group having a basis of open neighborhoods of 0 consisting of $m\widehat{\mathbb{Z}}$ for $m \in \mathbb{Q}$. Our first result, given with brief proof, is:

Theorem 1.1. *The Fuchsian group $\Delta(u^2, 2t)$ is pseudomodular or arithmetic if and only if its cusps (without ∞) are dense in the ring $\mathbb{A}_{\mathbb{Q},f}$ of finite adeles over $\mathbb{Q}$.*

Proof. The "only if" statement is trivial. To establish the converse, we note that $\Delta(u^2, 2t)$ contains the translation $z \mapsto z + 2t$ given by $g_1 g_2^{-1} g_1^{-1} g_2$. If $\Delta(u^2, 2t)$ is neither pseudomodular nor arithmetic, then since ∞ is a cusp, some rational x must not be a cusp. The orbit U of x under the translation $z \mapsto z + 2t$ is an open subset of $\mathbb{Q}$ given the subspace topology from $\mathbb{A}_{\mathbb{Q},f}$. This U contains no cusps, so the set of finite cusps of $\Delta(u^2, 2t)$ is not dense in $\mathbb{A}_{\mathbb{Q},f}$. □

Theorem 1.1 is in fact true for arbitrary zonal Fuchsian subgroups of $\mathrm{PSL}_2(\mathbb{Q})$. We remark that for a given t only finitely many u^2 yield arithmetic groups. We shall use refinements of Theorem 1.1 to give explicit, infinite families of $\Delta(u^2, 2t)$ whose cusp sets are proper subsets of $\mathbb{P}^1(\mathbb{Q})$. For example:

Theorem 1.2. *Let p be a prime and t an integer at least 2. Then $\Delta(p^{-2}, 2t)$ is neither pseudomodular nor arithmetic. The collection of groups $\Delta(p^{-2}, 2t)$ spans infinitely many commensurability classes; in particular, there are infinitely many commensurability classes of Fricke groups with rational cusps that are neither pseudomodular nor arithmetic.*

We will provide a proof for the first part of this theorem in Section 2.

In [5], Long and Reid exhibit finitely many $\Delta(u^2, 2t)$ that are neither pseudomodular nor arithmetic. For each such group, they provide a rational number fixed by a hyperbolic element of $\Delta(u^2, 2t)$. Such fixed points cannot be cusps; see the proof of Theorem 8.3.1 in [2]. We do not know whether rational hyperbolic fixed points exist for all non-pseudomodular $\Delta(u^2, 2t)$, and in any case, our proofs do not require or produce them.

Our results below involve the density of cusp sets in various topologies on $\mathbb{P}^1(\mathbb{Q})$. Each of these topologies is Hausdorff and we are only considering zonal Fuchsian groups, so density of the set of cusps in $\mathbb{P}^1(\mathbb{Q})$ is equivalent to density of finite cusps in $\mathbb{Q}$. Therefore the results below are comparable with Theorem 1.1.

2. Results

Denote by $\mathcal{C}(G)$ the cusp set of a Fuchsian group G. We are interested in the question of when $\mathcal{C}(\Delta(u^2, 2t))$ is $\mathbb{P}^1(\mathbb{Q})$. If $\mathcal{C}(\Delta(u^2, 2t))$ is not dense in some finite product $\prod_i \mathbb{P}^1(\mathbb{Q}_{p_i})$ with $\mathbb{P}^1(\mathbb{Q})$ embedded diagonally, then $\mathcal{C}(\Delta(u^2, 2t)) \neq \mathbb{P}^1(\mathbb{Q})$ since $\mathbb{P}^1(\mathbb{Q})$ is dense in said product.

For p a prime, we denote by v_p the p-adic valuation of $\mathbb{Q}_p$.

Proposition 2.1. *Let p be prime. If $v_p(t) \geqslant 0$ and $v_p(u^2) \leqslant -2$, or if $v_p(t) < 0$ and $v_p(u^2) \leqslant 2(v_p(t) - 1)$, then $\mathcal{C}(\Delta(u^2, 2t))$ is not dense in $\mathbb{P}^1(\mathbb{Q}_p)$.*

Proposition 2.2. *If p and q are prime, $v_p(u^2) = -1 = v_q(u^2)$, and t is p-adically and q-adically integral, then $\mathcal{C}(\Delta(u^2, 2t))$ is not dense in $\mathbb{P}^1(\mathbb{Q}_p) \times \mathbb{P}^1(\mathbb{Q}_q)$.*

Results similar to Proposition 2.2 hold for t that are not p-adically integral or not q-adically integral; we omit their statements for brevity. As a corollary to these two propositions, whenever t is an integer and the denominator of u^2 is composite, $\Delta(u^2, 2t)$ is not pseudomodular. We in fact have a stronger result:

Proposition 2.3. *Let t be an integer and suppose $\Delta(u^2, 2t)$ is pseudomodular. Then*

(a) *u^2 has prime or unit denominator, say p,*
(b) *if this p is an odd prime, then p does not divide t, and*
(c) *for all odd primes q dividing t, u^2 (necessarily in $\mathbb{Z}_q$) is congruent to 0 or $-1 \bmod q$.*

Here we prove the first part of Theorem 1.2 by establishing the first part of Proposition 2.1.

Proof. Let p be prime, let t be p-adically integral and select u^2 with $v_p(u^2) \leqslant -2$. Accordingly, write $u^2 = m/p^a$ with m a p-adic unit and a an integer at least 2. Suppose $x \in \mathbb{Q}$ with $v_p(u^2) < v_p(x) < 0$. Set $e = -v_p(x)$ and hence write $x = r/s$ with r and s coprime integers such that $p \nmid r$ and $p^e \parallel s$. Also, define $s_0 := s/p^e$. We compute $v_p(g_i^{\pm 1}x)$ $(i = 1, 2)$ by representing the elements $g_i^{\pm 1}$ by matrices in $\mathrm{PGL}_2(\mathbb{R})$ whose entries are all p-adic integers. For example:

$$v_p(g_1 x) = v_p\big((t-1)p^a r + ms\big) - v_p(p^a r + p^a s) = e + v_p\big((t-1)p^{a-e} r + ms_0\big) - a = e - a.$$

Similarly,

$$v_p(g_1^{-1}x) = e - a \quad \text{and} \quad v_p(g_2 x) = v_p(g_2^{-1}x) = -e.$$

We assumed that $-a < -e < 0$, so we have $-a < e - a < 0$. Therefore, $v_p(u^2) < v_p(g_i^{\pm 1}x) < 0$ for $i = 1, 2$. We conclude that $\Delta(u^2, 2t)$ leaves invariant the p-adically open, proper subset $\{x \in \mathbb{Q}\colon v_p(u^2) < v_p(x) < 0\}$ of $\mathbb{P}^1(\mathbb{Q})$. Since this set misses ∞, which generates the single orbit of cusps of $\Delta(u^2, 2t)$, $\mathcal{C}(\Delta(u^2, 2t))$ is not dense in $\mathbb{P}^1(\mathbb{Q}_p)$. □

Propositions 2.2 and 2.3 and the remainder of Proposition 2.1 are proved similarly by finding proper, non-empty $\Delta(u^2, 2t)$-invariant subsets of $\mathbb{P}^1(\mathbb{Q})$ that miss ∞ and that are open in the topology induced by that of $\mathbb{P}^1(\mathbb{Q}_p)$ or $\mathbb{P}^1(\mathbb{Q}_p) \times \mathbb{P}^1(\mathbb{Q}_q)$.

In contrast to the above results, there are many groups $\Delta(u^2, 2t)$ whose cusp sets are dense in *every* topology on $\mathbb{P}^1(\mathbb{Q})$ induced by a product of p-adic fields. For example, if t is prime, u^2 has prime denominator not equal to t and $u^2 \equiv 0$ or $-1 \bmod t$, then $\mathcal{C}(\Delta(u^2, 2t))$ is dense in the product $\prod_p \mathbb{P}^1(\mathbb{Q}_p)$ over all primes. There are groups with hyperbolic fixed points to which this statement applies, such as $\Delta(6/11, 6)$ with a hyperbolic fixed point of $1/4$. Consequently:

Theorem 2.4. *Let $\Delta(u^2, 2t)$ be non-arithmetic. Then the density of $\mathcal{C}(\Delta(u^2, 2t))$ in the product $\prod_p \mathbb{P}^1(\mathbb{Q}_p)$, ranging over all primes p and given the product topology, is not a sufficient condition for pseudomodularity.*

3. Questions

If the parameters u^2 and t are algebraic numbers and $K = \mathbb{Q}(u^2, t)$, then (as in [4]) we call $\Delta(u^2, 2t)$ *maximally cusped* if $\mathcal{C}(\Delta(u^2, 2t)) = \mathbb{P}^1(K)$. Maximally cusped groups can give information about the class group of K. For example, if a maximally cusped group $\Delta(u^2, 2t)$ is a subgroup of $\mathrm{PSL}_2(O_K)$ with $K = \mathbb{Q}(u^2, t)$, then the class number of K is one. Thus we are interested in finding necessary and sufficient conditions on u^2 and t for $\Delta(u^2, 2t)$ being maximally cusped.

Above, we considered the most basic case, with $K = \mathbb{Q}$, by finding obstructions to $\Delta(u^2, 2t)$ being maximally cusped (i.e., pseudomodular) using p-adic topologies on $\mathbb{P}^1(\mathbb{Q})$. By Theorem 2.4, our considerations are not enough to characterize pseudomodularity. One way to extend our work is to investigate density of cusps in topologies that strictly refine p-adic topologies on $\mathbb{P}^1(\mathbb{Q})$, such as the following:

Definition 3.1. Identify $\mathbb{P}^1(\mathbb{Q})$ with $\mathbb{P}^1(\mathbb{Z})$, understood as a subset of $\mathbb{Z}^2/\{\pm 1\}$. Let S be the set of all primes (resp., a finite set of primes). The diagonal embedding of $\mathbb{Z}^2/\{\pm 1\}$ in $(\prod_{p \in S} \mathbb{Z}_p^2)/\{\pm 1\}$, endowed with the product topology, induces a topology on $\mathbb{P}^1(\mathbb{Q})$ which we call the *congruence topology* (resp., the *S-congruence topology*).

Since the congruence topology is finer than that induced by the product $\prod_p \mathbb{P}^1(\mathbb{Q}_p)$ over all primes, the cusp sets of the groups of Propositions 2.1 and 2.2 are not dense in $\mathbb{P}^1(\mathbb{Q})$ given the congruence topology. We also have examples of groups whose cusp sets are not dense in the congruence topology on $\mathbb{P}^1(\mathbb{Q})$ despite being dense in the topology induced by the product $\prod_p \mathbb{P}^1(\mathbb{Q}_p)$ over all primes. We give such an example now.

Let $\Lambda(u^2, 2t)$ be the kernel of the group homomorphism $\Delta(u^2, 2t) \to \mathbb{Z}/2 \oplus \mathbb{Z}/2$ given by $g_1 \mapsto (1, 0)$ and $g_2 \mapsto (0, 1)$. Then $\Lambda(u^2, 2t) \subseteq \mathrm{PSL}_2(\mathbb{Q})$ and $\mathcal{C}(\Lambda(u^2, 2t)) = \mathcal{C}(\Delta(u^2, 2t))$. Consider the group $\Delta := \Delta(6/11, 6)$. We can construct a subgroup K of $\mathrm{PSL}_2(\mathbb{Z}[2^{-1}])$ containing $\Lambda(6/11, 6)$ that has eight orbits in its action on $\mathbb{P}^1(\mathbb{Q})$. An explicit description of these K-orbits gives us a Δ-invariant, non-empty, proper subset X of $\mathbb{P}^1(\mathbb{Q})$ that is open in the S-congruence topology for $S = \{3, 11\}$ and hence open in the congruence topology on $\mathbb{P}^1(\mathbb{Q})$. By the remarks immediately prior to Theorem 2.4, the cusp set of Δ is nevertheless dense in $\prod_p \mathbb{P}^1(\mathbb{Q}_p)$, and Δ is neither pseudomodular nor arithmetic. This motivates the following

Question. Suppose that the set $\mathcal{C}(\Delta(u^2, 2t))$ is dense in the congruence topology on $\mathbb{P}^1(\mathbb{Q})$, or equivalently if, for every integer $N > 0$, the image of $\mathcal{C}(\Delta(u^2, 2t))$ in $(\mathbb{Z}/N\mathbb{Z})^2/\{\pm 1\}$ consists of all classes of elements of order N. Is the group $\Delta(u^2, 2t)$ pseudomodular or arithmetic?

References

[1] R. Abe, On correspondences between once punctured tori and closed tori: Fricke groups and real lattices, Tokyo J. Math. 23 (2) (2000) 269–293.

[2] A.F. Beardon, The Geometry of Discrete Groups, Graduate Texts in Math., vol. 91, Springer-Verlag, 1983.

[3] C. Culler, P. Shalen, Varieties of group representations and splittings of 3-manifolds, Ann. of Math. 117 (1) (1983) 109–146.

[4] D.D. Long, A.W. Reid, Generalized Dedekind sums, in: Proceedings of the Casson Fest, in: Geometry and Topology Monographs, vol. 7, 2004, pp. 205–212.

[5] D.D. Long, A.W. Reid, Pseudomodular surfaces, J. Reine Angew. Math. 552 (2002) 77–100.

[6] G. Shimura, Introduction to the Arithmetic Theory of Automorphic Functions, Princeton University Press, 1971.