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Homological Algebra

A derived functor approach to bounded cohomology

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Abstract

We apply the theory of the derived category of exact categories to the category G–**Ban** of Banach modules over the discrete group G. Since there are enough injectives in G–**Ban**, right derived functors exist. The heart of the canonical t-structure on the derived category **D**(**Ban**) is equivalent to Waelbroeck's Abelian category **qBan** of quotient Banach spaces. The right derived functor of the functor "submodule of G-invariant vectors" yields a universal δ-functor with values in **qBan** which allows us to reconstruct the bounded cohomology functors in the sense of Gromov–Brooks–Ivanov–Noskov. ***To cite this article: T. Bühler, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

La cohomologie bornée via foncteurs dérivés. Nous appliquons la théorie des catégories dérivées des catégories exactes à la catégorie G–**Ban** des modules de Banach du groupe discret G. Comme il y a assez d'injectifs dans G–**Ban**, les foncteurs dérivés à droite existent. Le cœur de la t-structure canonique dans la catégorie dérivée **D**(**Ban**) est équivalent à la catégorie abélienne **qBan** des espaces quotients banachiques au sens de Waelbroeck. En dérivant à droite le foncteur « sous-module des vecteurs G-invariants », nous obtenons un δ-foncteur universel à valeurs dans **qBan**, ce qui nous permet de reconstruire le foncteur de cohomologie bornée au sens de Gromov–Brooks–Ivanov–Noskov. ***Pour citer cet article : T. Bühler, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

1. Introduction

Bounded cohomology for spaces was introduced by Gromov [3] in the late seventies in order to define topological invariants which control the minimal volume of a smooth manifold. The most famous instance is the simplicial volume. One of Gromov's rather deep theorems is that the bounded cohomology of a connected countable CW-complex is an invariant of its fundamental group. To make this precise, Gromov relies on unpublished results by Trauber concerning a "cohomology theory" for groups with values in semi-normed spaces. This cohomology theory was further developed in the spirit of Hochschild–Mostow [4] by Brooks [2], Ivanov [5] and Noskov [10]. A nice and up-to-date account is given in Löh's thesis [8].

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Instead of painfully reproving the basic lemmas of homological algebra in a functional-analytic context, we propose to use a more modern approach by appealing to the theory of exact categories in the sense of Quillen [12] (see also [6, Appendix A], [14, Appendix A]) and their derived categories as discussed in [9] and [7] (see also [1]).

2. Preliminaries

Let **Ban** be the additive category of Banach spaces and bounded linear maps over the field k of real or complex numbers. It is a well-known consequence of the open mapping theorem that the class of all kernel-cokernel pairs is an exact structure in the sense of Quillen [12, § 2], see e.g. [1, Exemple 2, p. 20]. In fact, **Ban** has kernels and cokernels and is a quasi-Abelian category in the sense of Prosmans [11] and Schneiders [13], see also Yoneda [16]. However, no non-trivial infinite limits and colimits exist in **Ban**.

For every set S, the space $\ell^1(S)$ of summable sequences indexed by S is projective in **Ban** and the space $\ell^\infty(S)$ of bounded sequences is injective (Hahn–Banach). Since every Banach space E is a quotient of $\ell^1(B_{\leqslant 1}(E))$ and a subspace of $\ell^\infty(B_{\leqslant 1}(E^*))$, the category **Ban** has enough projectives and injectives.

Let G be a discrete group and let G–**Ban** be the category of isometric representations of G on Banach spaces (Banach G-modules) and bounded G-equivariant maps. Again, the class of all kernel-cokernel pairs is an exact structure in the sense of Quillen and G–**Ban** is quasi-Abelian. By direct inspection one obtains:

Theorem 2.1 *(Fundamental Adjunctions). Let* $\downarrow: G\text{–}\mathbf{Ban} \to \mathbf{Ban}$ *be the forgetful functor and let* ${}_\varepsilon(-) : \mathbf{Ban} \to G\text{–}\mathbf{Ban}$ *be the trivial module functor* (*augmentation*). *The forgetful functor has a left adjoint given by induction* $\ell^1(G)\widehat{\otimes}-$ *and a right adjoint given by coinduction* $\mathrm{Hom}_{\mathbf{Ban}}(\ell^1(G), -)$. *The trivial module functor has a left adjoint* $(-)_G = k\widehat{\otimes}_G-$ *and a right adjoint* $(-)^G = \mathrm{Hom}_{G\text{–}\mathbf{Ban}}(k, -)$.

The forgetful functor and the trivial module functor have adjoints on both sides, hence they are both exact. Induction and coinduction are exact since $\ell^1(G)$ is projective and hence flat as a Banach space.

Corollary 2.2. *There are enough projectives and injectives in* G–**Ban**.

Indeed, let M be a Banach G-module. The counit of the adjunction of induction and the forgetful functor yields an admissible epic $\ell^1(G)\widehat{\otimes}\downarrow M \twoheadrightarrow M$ in G–**Ban**. Now choose a projective presentation $P \twoheadrightarrow \downarrow M$. Since induction is left adjoint to an exact functor, it preserves projectives and cokernels and hence $\ell^1(G)\widehat{\otimes}P \twoheadrightarrow \ell^1(G)\widehat{\otimes}\downarrow M \twoheadrightarrow M$ exhibits M as a quotient of a projective Banach G-module. The case of injectives is dual.

Corollary 2.2 allows us to construct left and right derived functors on G–**Ban** using:

Corollary 2.3. *Let* $\mathscr{P}(G)$ *and* $\mathscr{I}(G)$ *be the classes of projective and injective objects in* G–**Ban**. *The homotopy category* $\mathbf{K}^-(\mathscr{P}(G))$ *of right bounded complexes of projective Banach* G*-modules is triangle equivalent to the derived category of right bounded complexes* $\mathbf{D}^-(G\text{–}\mathbf{Ban})$ *of Banach* G*-modules. Similarly,* $\mathbf{K}^+(\mathscr{I}(G))$ *is triangle equivalent to* $\mathbf{D}^+(G\text{–}\mathbf{Ban})$.

In particular we have the following result concerning functors defined on G–**Ban**:

Corollary 2.4. *Left derived functors exist and are defined on* $\mathbf{D}^-(G\text{–}\mathbf{Ban})$ *and right derived functors exist and are defined on* $\mathbf{D}^+(G\text{–}\mathbf{Ban})$.

3. The canonical t-structures

As in [1, 1.3.22] we introduce the (left) truncation functors $\tau_\ell^{\leqslant n}$ and $\tau_\ell^{\geqslant n}$ on $\mathbf{D}(G\text{–}\mathbf{Ban})$ given by

$$\tau_\ell^{\leqslant n} A^\bullet = (\cdots \to A^{n-2} \to A^{n-1} \to \operatorname{Ker} d^n \to 0 \to \cdots)$$

and

$$\tau_\ell^{\geqslant n} A^\bullet = (\cdots \to 0 \to \operatorname{Coim} d^{n-1} \to A^n \to A^{n+1} \to \cdots).$$

The (right) truncation functors $\tau_r^{\leqslant n}$ and $\tau_r^{\geqslant n}$ are defined dually. Let $\mathbf{D}_\ell^{\leqslant n}(G)$ be the essential image of $\tau_\ell^{\leqslant n}$, etc. In [1, 1.3.22] it is shown that $(\mathbf{D}_\ell^{\leqslant 0}(G), \mathbf{D}_\ell^{\geqslant 0}(G))$ is a non-degenerate t-structure. Its heart $\mathscr{C}_\ell(G) = \mathbf{D}_\ell^{\leqslant 0}(G) \cap \mathbf{D}_\ell^{\geqslant 0}(G)$ is Abelian. It is equivalent to Waelbroeck's Abelian category **qBan** (see [15]) in case G is the trivial group (this follows immediately from [15, 2.1.25] and the description below).

There is a rather explicit description of $\mathscr{C}_\ell(G)$ (see [1, 1.3.22]): the objects are represented by the monomorphisms $(A^{-1} \hookrightarrow A^0)$ in G–**Ban**, while the morphisms are obtained from the morphisms of pairs by dividing out the homotopy equivalence relation and by inverting quasi-isomorphisms (bicartesian diagrams) formally.

Theorem 3.1. *The inclusion functor G–**Ban** $\to \mathscr{C}_\ell(G)$ given on objects by $M \mapsto (0 \hookrightarrow M)$ is fully faithful, exact, reflects exactness and preserves monics. It admits a left adjoint since G–**Ban** has cokernels. Every exact and monic-preserving functor $F : G$–**Ban** $\to \mathscr{A}$ to an Abelian category $\mathscr{A}$ factors uniquely through an exact functor $\mathscr{C}_\ell(G) \to \mathscr{A}$.*

By [1, 1.3.23(iii)] and [13, 1.2.32] we have:

Theorem 3.2. *The inclusion functor extends to triangle equivalences $\mathbf{D}^-(G\text{–}\mathbf{Ban}) \cong \mathbf{D}^-(\mathscr{C}_\ell(G))$ and $\mathbf{D}(G\text{–}\mathbf{Ban}) \cong \mathbf{D}(\mathscr{C}_\ell(G))$.*

Let **Csn** be the category of complete seminormed vector spaces and continuous linear maps, where completeness is understood in the sense that every Cauchy sequence has an accumulation point. Notice that the Hausdorffification of a complete seminormed space is a Banach space.

Lemma 3.3. *The functor $real : \mathscr{C}_\ell(G) \to \mathbf{Csn}$ given on objects by $\mathrm{Coker}_{\mathbf{Csn}}(A^{-1} \hookrightarrow A^0)$ is well-defined and transforms exact sequences in $\mathscr{C}_\ell(G)$ to sequences whose underlying sequence of vector spaces is exact.*

4. The main results

The right derived functor $\mathbf{R}^+(-)^G : \mathbf{D}^+(G\text{–}\mathbf{Ban}) \to \mathbf{D}^+(\mathbf{Ban})$ exists by Corollary 2.2 and is a triangle functor by definition, see [7, §§13–15]. By composing $\mathbf{R}^+(-)^G$ with the homological functor $H^0 : \mathbf{D}^+(\mathbf{Ban}) \to \mathbf{qBan}$, we obtain a δ-functor $\mathscr{H}^n(G,-) : G$–**Ban** $\to$ **qBan**. Moreover, we have:

Theorem 4.1. *Up to unique isomorphism of δ-functors there is a unique family of functors*

$$\mathscr{H}^n(G,-) : G\text{–}\mathbf{Ban} \longrightarrow \mathbf{qBan}, \quad n \in \mathbb{Z},$$

having the following properties:

(i) (*Normalization*) *$\mathscr{H}^0(G,M) = (0 \hookrightarrow M^G)$ for all $M \in G$–**Ban**.*
(ii) (*Vanishing*) *$\mathscr{H}^n(G,I) = 0$ for all injective objects I in G–**Ban** and all $n > 0$.*
(iii) (*Long exact sequence*) *For every short exact sequence $M' \rightarrowtail M \twoheadrightarrow M''$ in G–**Ban** there is a family of connecting morphisms $\mathscr{H}^n(G,M'') \to \delta^n \mathscr{H}^{n+1}(G,M')$ depending naturally on the sequence and fitting into a long exact sequence*

$$\cdots \xrightarrow{\delta^{n-1}} \mathscr{H}^n(G,M') \to \mathscr{H}^n(G,M) \to \mathscr{H}^n(G,M'') \xrightarrow{\delta^n} \mathscr{H}^{n+1}(G,M') \to \cdots$$

in **qBan**.

Let $\mathscr{H}_b^n(G,-) : G$–**Ban** $\to$ **Csn** be the composition real $\circ\, \mathscr{H}^n(G,-)$. Obviously Lemma 3.3 implies:

Corollary 4.2. *There is a family of functors*

$$\mathscr{H}_b^n(G,-) : G\text{–}\mathbf{Ban} \longrightarrow \mathbf{Csn}, \quad n \in \mathbb{Z},$$

having the following properties:

(i) (*Normalization*) $\mathscr{H}_b^0(G,M) = M^G$ *for all* $M \in G$–**Ban**.
(ii) (*Vanishing*) $\mathscr{H}_b^n(G,I) = 0$ *for all injective objects* I *in* G–**Ban** *and all* $n > 0$.
(iii) (*Long exact sequence*) *For every short exact sequence* $M' \rightarrowtail M \twoheadrightarrow M''$ *in* G–**Ban** *there is a family of connecting morphisms* $\mathscr{H}_b^n(G,M'') \xrightarrow{\delta^n} \mathscr{H}_b^{n+1}(G,M')$ *depending naturally on the sequence and fitting into a long sequence*

$$\cdots \xrightarrow{\delta^{n-1}} \mathscr{H}_b^n(G,M') \to \mathscr{H}_b^n(G,M) \to \mathscr{H}_b^n(G,M'') \xrightarrow{\delta^n} \mathscr{H}_b^{n+1}(G,M') \to \cdots$$

in **Csn** *whose underlying sequence of vector spaces is exact.*

Theorem 4.3. *The functors* $\mathscr{H}_b^n(G,-)$ *are isomorphic to the bounded cohomology functors* $\widehat{H}^n(G,-)$ *described in* [3,2,5,10].

References

[1] A.A. Beĭlinson, J. Bernstein, P. Deligne, Faisceaux pervers, in: Analysis and Topology on Singular Spaces, I, Luminy, 1981, in: Astérisque, vol. 100, Soc. Math. France, Paris, 1982, pp. 5–171. MR751966 (86g:32015).

[2] R. Brooks, Some remarks on bounded cohomology, in: Riemann Surfaces and Related Topics: Proceedings of the 1978 Stony Brook Conference, State Univ. New York, Stony Brook, NY, 1978, in: Ann. of Math. Stud., vol. 97, Princeton Univ. Press, Princeton, NJ, 1981, pp. 53–63, MR624804 (83a:57038).

[3] M. Gromov, Volume and bounded cohomology, Inst. Hautes Études Sci. Publ. Math. 56 (1982) 5–99. MR686042 (84h:53053).

[4] G. Hochschild, G.D. Mostow, Cohomology of Lie groups, Illinois J. Math. 6 (1962) 367–401. MR0147577 (26 #5092).

[5] N.V. Ivanov, Foundations of the theory of bounded cohomology, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 143 (1985) 69–109, 177–178. MR806562 (87b:53070).

[6] B. Keller, Chain complexes and stable categories, Manuscripta Math. 67 (4) (1990) 379–417. MR1052551 (91h:18006).

[7] B. Keller, Derived categories and their uses, in: Handbook of Algebra, vol. 1, North-Holland, Amsterdam, 1996, pp. 671–701, MR1421815 (98h:18013).

[8] C. Löh, ℓ^1-homology and simplicial volume, Ph.D. thesis, Westfälische Wilhelms-Universität Münster, 2007.

[9] A. Neeman, The derived category of an exact category, J. Algebra 135 (2) (1990) 388–394. MR1080854 (91m:18016).

[10] G.A. Noskov, Bounded cohomology of discrete groups with coefficients, Algebra i Analiz 2 (5) (1990) 146–164. MR1086449 (92b:57005).

[11] F. Prosmans, Derived categories for functional analysis, Publ. Res. Inst. Math. Sci. 36 (1) (2000) 19–83. MR1749013 (2001g:46156).

[12] D. Quillen, Higher algebraic K-theory. I, in: Algebraic K-Theory, I: Higher K-Theories, Proc. Conf., Battelle Memorial Inst., Seattle, WA, 1972, in: Lecture Notes in Math., vol. 341, Springer, Berlin, 1973, pp. 85–147. MR0338129 (49 #2895).

[13] J.-P. Schneiders, Quasi-abelian categories and sheaves, Mém. Soc. Math. France (N.S.) 76 (1999) vi+134. MR1779315 (2001i:18023).

[14] R.W. Thomason, T. Trobaugh, Higher algebraic K-theory of schemes and of derived categories, in: The Grothendieck Festschrift, vol. III, in: Progr. Math., vol. 88, Birkhäuser Boston, Boston, MA, 1990, pp. 247–435. MR1106918 (92f:19001).

[15] L. Waelbroeck, Bornological quotients, Mémoire de la Classe des Sciences. Collection in-4°. 3° Série (Memoir of the Science Section. Collection in-4°. 3rd Series), VII, Académie Royale de Belgique. Classe des Sciences, Brussels, 2005, With the collaboration of Guy Noël. MR2128718 (2006b:46001).

[16] N. Yoneda, On Ext and exact sequences, J. Fac. Sci. Univ. Tokyo Sect. I 8 (1960) 507–576. MR0225854 (37 #1445).