Complete gradient shrinking Ricci solitons have finite topological type

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Abstract
We show that a complete Riemannian manifold has finite topological type (i.e., homeomorphic to the interior of a compact manifold with boundary), provided its Bakry–Émery Ricci tensor has a positive lower bound, and either of the following conditions:
(i) the Ricci curvature is bounded from above;
(ii) the Ricci curvature is bounded from below and injectivity radius is bounded away from zero.
Moreover, a complete shrinking Ricci soliton has finite topological type if its scalar curvature is bounded.

1. Introduction
In 1968, J. Milnor [8] conjectured that a complete non-compact Riemannian manifold with non-negative Ricci curvature has a finitely generated fundamental group. However, such a manifold may not have finite topological type. Examples of complete non-compact manifold with positive Ricci curvature without finite topological type was constructed by Gromoll–Meyer [5]. It has been an interesting topic in Riemannian geometry to study the topology of complete manifolds with positive (non-negative) Ricci curvature.
In this Note we are concerned with complete Riemannian manifold \((M, g)\) satisfying that \(\text{Ric} + \text{Hess}(f) \geq \lambda g\) for some constant \(\lambda > 0\) and \(f \in C^\infty(M)\), i.e., whose Bakry–Émery Ricci tensor is bounded below by \(\lambda\) in the sense of [7]. When the equality holds, the manifold is a shrinking Ricci soliton, i.e., a self-similar solution of the well-known Ricci flow equation. If \(f\) is constant, Bakry–Émery Ricci tensor reduces to the Ricci tensor, and so the classical Myers’ theorem implies that \(M\) is compact with finite fundamental group. In general, \(M\) may not be compact, but from the work of [1,3,4,6,7,10,9,11] etc., \(M\) still has finite fundamental group. Under some additional conditions, the manifold may be of finite topological type:

**Theorem 1.1.** Suppose \((M, g)\) is a complete Riemannian manifold satisfying \(\text{Ric} + \text{Hess}(f) \geq \lambda g\) for some constant \(\lambda > 0\) and \(f \in C^\infty(M)\). Then \(M\) is of finite topological type, if either of the following alternative conditions holds:

(i) \(\text{Ric} \leq C g\) for some constant \(C < \infty\);

(ii) \(\text{Ric} \geq -\delta^{-1} g\) and the injectivity radius \(\text{inj}(M, g) \geq \delta > 0\) for some \(\delta > 0\).

If \((M, g)\) is a shrinking Ricci soliton, then the Ricci curvature bounds can be relaxed by scalar curvature.

**Theorem 1.2.** Suppose \((M, g)\) is a complete shrinking Ricci soliton \(\text{Ric} + \text{Hess}(f) = \frac{8}{7} g\), where \(f \in C^\infty(M)\). If the scalar curvature \(R\) is bounded, then \(M\) has finite topological type.

In view of Theorem 1.2 it is natural to pose the following:

**Conjecture 1.3.** Any shrinking Ricci soliton has finite topological type.

We prove Theorem 1.1 in Section 2 and Theorem 1.2 in Section 3.

### 2. Proof of Theorem 1.1

Let \((M, g)\) be such a manifold satisfying that \(\text{Ric} + \text{Hess}(f) \geq \lambda g\) for some \(\lambda > 0\) and \(f \in C^\infty(M)\). By the deformation lemma of Morse theory, to prove Theorem 1.1, it suffices to show that the function \(f\) is proper and has no critical points outside of a compact set.

First fix one point \(p \in M\) as a base point. For any \(q \in M\) with \(d(p, q) = L\), choose a shortest geodesic \(\gamma\) from \(p\) to \(q\) parametrized by arc length. Then

\[
\langle \nabla f, \dot{\gamma} \rangle(q) = \langle \nabla f, \dot{\gamma} \rangle(p) + \int_0^L \frac{d^2}{dt^2} f(\gamma(t)) \, dt = \langle \nabla f, \dot{\gamma} \rangle(p) + \int_0^L (\lambda - \text{Ric}(\dot{\gamma}, \dot{\gamma})) \, dt
\]

\[
\geq \lambda L - |\nabla f|(p) - \int_0^L \text{Ric}(\dot{\gamma}, \dot{\gamma}) \, dt.
\]

If the integral \(\int_0^L \text{Ric}(\dot{\gamma}, \dot{\gamma}) \, dt \leq \Lambda\) for some constant \(\Lambda\) independent of \(q\) and the choice of \(\gamma\), then \(|\nabla f|(q) \geq \langle \nabla f, \dot{\gamma} \rangle(q) \geq \lambda d(p, q) - |\nabla f|(p) - \Lambda\), which implies that \(|\nabla f|(q)\) has a linear growth in \(d(p, q)\) and so \(f\) is a proper function without critical points outside of a compact set. In the remainder of this section, we will focus on proving that \(\int_0^L \text{Ric}(\dot{\gamma}, \dot{\gamma}) \, dt \leq \Lambda\) has an upper bound under the assumptions of Theorem 1.1.

**Case (i):** \(\text{Ric} \leq C g\) for some constant \(C < \infty\). By Lemma 2.2 of [10], the integral bound is given by \(\Lambda = 2(n - 1) + 2C\).

**Case (ii):** \(\text{Ric} \geq -\delta^{-1} g\) and \(\text{inj}(M, g) \geq \delta > 0\) for some \(\delta > 0\). Suppose \(d(p, q) = L \geq \delta\). Let \(\varphi(t) : [0, L] \to [0, 1]\) be an arcwise smooth function such that \(\varphi(0) = \varphi(L) = 0\). By the second variation formula, as proposed in [10] or [11], we have the estimate: \(\int_0^L \varphi^2(t) \text{Hess}(f)(\dot{\gamma}, \dot{\gamma}) \, dt \leq (n - 1) \int_0^L |\varphi'|^2 \, dt\). Now define \(\varphi\) by

\[
\varphi(t) = \begin{cases} \frac{3}{\delta} t, & t \in [0, \frac{\delta}{3}] \\ 1, & t \in \left[\frac{\delta}{3}, L - \frac{\delta}{3}\right] \\ \frac{3}{\delta}(L - t), & t \in \left[L - \frac{\delta}{3}, L\right]. \end{cases}
\]
then we have the estimate

$$
\int_{0}^{L} \text{Ric}(\dot{\gamma}, \dot{\gamma}) \, dt \leq (n - 1) \int_{0}^{L} |\dot{\psi}|^{2} \, dt + \frac{\delta^{3}}{L} \int_{0}^{\delta/3} (1 - \varphi^{2}) \text{Ric}(\dot{\gamma}, \dot{\gamma}) \, dt + \int_{L - \delta/3}^{L} (1 - \varphi^{2}) \text{Ric}(\dot{\gamma}, \dot{\gamma}) \, dt
$$

\[
\leq \frac{6}{\delta} (n - 1) + \frac{2}{3} + \frac{\delta^{3}}{L} \int_{0}^{\delta/3} \text{Ric}(\dot{\gamma}, \dot{\gamma}) \, dt + \frac{\delta^{3}}{L} \int_{L - \delta/3}^{L} \text{Ric}(\dot{\gamma}, \dot{\gamma}) \, dt,
\]

where in the second inequality, we used the fact that

$$
- \int_{0}^{\delta/3} \varphi^{2} \text{Ric}(\dot{\gamma}, \dot{\gamma}) \, dt \leq \frac{1}{\delta} \int_{0}^{\delta/3} \varphi^{2} \, dt = \frac{1}{9}, \quad \text{and similarly}, \quad - \int_{L - \delta/3}^{L} \varphi^{2} \text{Ric}(\dot{\gamma}, \dot{\gamma}) \, dt \leq \frac{1}{9}.
$$

We next prove that \( \int_{0}^{\delta/3} \text{Ric}(\dot{\gamma}, \dot{\gamma}) \, dt \) and \( \int_{L - \delta/3}^{L} \text{Ric}(\dot{\gamma}, \dot{\gamma}) \, dt \) are bounded from above and so finish the proof of Theorem 1.1. This is given by the following lemma:

**Lemma 2.1.** If \( \text{Ric} \geq -\delta^{-1} g \) and inj\((M, g) \geq \delta > 0 \) for some \( \delta > 0 \), then

$$
\int_{0}^{\delta/3} \text{Ric}(\dot{\gamma}, \dot{\gamma}) \, dt \leq \frac{6}{\delta} (n - 1) + \frac{2}{3}
$$

for any minimal arc length parametrized geodesic \( \gamma : [0, \delta] \to M \).

**Proof.** Firstly, by inj\((M, g) \geq \delta \), we can extend the geodesic \( \gamma \) to a shortest geodesic \( \sigma : [0, \delta] \to M \), such that \( \gamma(t) = \sigma(t + \delta/2), t \in [0, \delta] \). Set \( L = \delta \) in the arguments above, we have \( \int_{0}^{\delta} \varphi^{2} \text{Ric}(\dot{\sigma}, \dot{\sigma}) \, dt \leq (n - 1) \int_{0}^{\delta} |\dot{\psi}|^{2} \, dt = \frac{6}{\delta} (n - 1) \), then using \( \text{Ric} \geq -\delta^{-1} g \), we get the estimate

$$
\int_{0}^{\delta/3} \text{Ric}(\dot{\gamma}, \dot{\gamma}) \, dt = \int_{0}^{\delta/3} \text{Ric}(\dot{\sigma}, \dot{\sigma}) \, dt \leq \frac{6}{\delta} (n - 1) - \int_{0}^{\delta/3} \varphi^{2} \text{Ric}(\dot{\sigma}, \dot{\sigma}) \, dt - \int_{\delta/3}^{\delta} \varphi^{2} \text{Ric}(\dot{\sigma}, \dot{\sigma}) \, dt
$$

\[
\leq \frac{6}{\delta} (n - 1) + \frac{2}{3}. \quad \square
\]

### 3. Proof of Theorem 1.2

As before, we will prove that the potential function \( f \) to the Ricci soliton is proper and has no critical points outside of \( B(p, \rho) \) for large \( \rho \). Suppose \((M, g) \) is a complete shrinking Ricci soliton which satisfies \( \text{Ric} + \text{Hess}(f) = \frac{2}{\rho} \) for some potential function \( f \). Suppose further that the scalar curvature \( |R| \leq C \) for some constant \( C < \infty \). It’s well-known that the following analytic equality holds for the soliton (after modifying \( f \) by a translation, see [2] for example):

$$
R + |\nabla f|^{2} = f.
$$

(1)

Let \( q \in M \) be one point and denote by \( \rho = d(p, q) \) the distance from \( p \) to \( q \). Let \( \gamma \) be a shortest arc length parametrized geodesic from \( p \) to \( q \). First we have:

**Lemma 3.1.** \( \frac{\rho}{2} - |\nabla f|(p) - |\nabla f|(q) \leq \int_{0}^{\rho} \text{Ric}(\dot{\gamma}, \dot{\gamma}) \, dt \).

**Proof.** It follows from a direct computation,

$$
|\nabla f|(q) \geq \langle \nabla f, \dot{\gamma}(q) \rangle = \langle \nabla f, \dot{\gamma}(p) \rangle + \int_{0}^{\rho} \frac{d^{2}}{dt^{2}} f(\gamma(t)) \, dt \geq -|\nabla f|(p) + \int_{0}^{\rho} \left(\frac{1}{2} - \text{Ric}(\dot{\gamma}, \dot{\gamma})\right) \, dt. \quad \square$$
On the other hand, by second variation formula as did in above section, we can get an upper bound of \( \int_0^\rho \text{Ric}(\dot{\gamma}, \dot{\gamma}) \, dt \). Precisely, for the function \( \psi(t) \) defined by

\[
\psi(t) = t, \quad t \in [0, 1]; \quad \psi(t) = 1, \quad t \in [1, \rho - 1]; \quad \psi(t) = \rho - t, \quad t \in [\rho - 1, \rho],
\]

we have the estimate

\[
\int_0^\rho \text{Ric}(\dot{\gamma}, \dot{\gamma}) \, dt \leq \int_0^\rho (n - 1)|\dot{\psi}|^2 \, dt + \int_0^1 (1 - \psi^2) \text{Ric}(\dot{\gamma}, \dot{\gamma}) \, dt + \int_0^\rho (1 - \psi^2) \text{Ric}(\dot{\gamma}, \dot{\gamma}) \, dt \\
\leq 2(n - 1) + \sup_{B(\rho, 1)} |\text{Ric}| + \int_0^\rho (1 - \psi^2) \left( \frac{1}{2} - \frac{d^2}{dt^2} f(\gamma(t)) \right) \, dt.
\]

By integration by parts, we have the estimate for the last term

\[
\int_0^\rho (1 - \psi^2) \left( \frac{1}{2} - \frac{d^2}{dt^2} f(\gamma(t)) \right) \, dt \leq 1 + 2 \int_0^\rho \psi \frac{d}{dt} f(\gamma(t)) \, dt = 1 - 2 f(\gamma(\rho - 1)) + 2 \int_0^\rho f(\gamma(t)) \, dt.
\]

Substituting this equality into above estimate, we obtain:

\[
\int_0^\rho \text{Ric}(\dot{\gamma}, \dot{\gamma}) \, dt \leq 2n + \sup_{B(\rho, 1)} |\text{Ric}| + 2 \int_0^\rho f(\gamma(t)) \, dt - 2 f(\gamma(\rho - 1)) \\
\leq 2n + \sup_{B(\rho, 1)} |\text{Ric}| + \sup_{x, y \in B(q, 1)} 2|f(x) - f(y)|.
\]

(2)

Now we are ready to show \( f \) is proper. By assumption \( |R| \leq C \) and then using Eq. (1), we see that \( |\nabla f| \leq \sqrt{f - R} \leq \sqrt{f} + C \). It follows that \( |\sqrt{f(x)} + C - \sqrt{f(q) + C}| \leq \frac{1}{2} \) for all \( x \in B(q, 1) \), and so

\[
|f(x) - f(y)| = |\sqrt{f(x)} + C + \sqrt{f(y)} + C| \cdot |\sqrt{f(x)} + C - \sqrt{f(y) + C}| \leq 2\sqrt{f(q) + C} + 1, \\
\forall x, y \in B(q, 1).
\]

The combination of Lemma 3.1, Eq. (2) and above estimate gives

\[
\frac{\rho}{2} \leq 2n + |\nabla f|(p) + |\nabla f|(q) + \sup_{B(\rho, 1)} |\text{Ric}| + 4\sqrt{f(q) + C} + 2 \\
\leq 2(n + 1) + |\nabla f|(p) + \sup_{B(\rho, 1)} |\text{Ric}| + 5\sqrt{f(q) + C}.
\]

It concludes that \( f \) is a proper function and then by boundedness of \( R \) and Eq. (1), \( f \) has no critical points outside of a compact domain. This finishes the proof of the theorem.

References