Number Theory

On binary palindromes of the form $10^n \pm 1$ ★

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Abstract

In this Note, we find all positive integers $n$ such that $10^n \pm 1$ is a binary palindrome. Our proof uses lower bounds for linear forms in logarithms of rational numbers. To cite this article: F. Luca, A. Togbé, C. R. Acad. Sci. Paris, Ser. I 346 (2008). © 2008 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

Résumé


1. Introduction

Let $b \geq 1$ be an integer. A base $b$ palindrome is a positive integer whose string of digits when written in base $b$ reads the same from left to right as from right to left. That is, if the base $b$ representation of $n$ is $n = \overline{a_1a_2\cdots a_m}_b$, then $a_i = a_{m-i+1}$ holds for all $i = 1, \ldots, m$. For example, $123454321$ is a base 10 palindrome, and $7 = \overline{111}_2$ is a binary palindrome. Palindromes are fascinating numbers to mathematicians and aficionados alike who find a lot of joy when accidentally stumbling upon a palindrome in real life. The construction of the famous Charles Bridge over the Vltava river was started by King Charles IV at 5:31am on the 9th of July, 1357 this minute being then enumerated as $135\,797\,531$. It seems to be very difficult to prove any strong results about the presence of palindromes in various arithmetically interesting sequences. For example, it is not known if there are infinitely many prime palindromes although this is conjectured to be so. In fact, it is not even known if there are infinitely many square-free palindromes but it is known that there are infinitely many cube free ones! Several interesting results about prime factors of palindromes can be found in [2] and [3]. Some recreational results about palindromic squares can be found in [1] and [4].
Given any positive integer $n$, both $10^n + 1 = 100 \ldots 01_{10}$ (with $n − 1$ zeros), as well as $10^n − 1 = 99 \ldots 9_{10}$ (with $n$ nines) are palindromes in base 10. In this Note, we look at the values of $n$ such that $10^n \pm 1$ is a base 2 palindrome.

In [6], it was shown that the set of positive integers $n$ such that $F_n$, the $n$th Fibonacci number, is a base $b$ palindrome is of asymptotic density zero. Almost certainly this set is finite for any given $b$, but this has not been proved yet. The result proved there in fact applies with the Fibonacci sequence $(F_n)_{n \geq 0}$ replaced by any Lucas sequence whose characteristic roots are quadratic units. In the last section of [6] it is shown how one may adapt the arguments used there to deal also with sequences of the form $a^n \pm 1$, where $a$ and $b$ multiplicatively independent. The condition that $a$ and $b$ are multiplicatively independent is needed for otherwise one can easily show that the set of $n$ such that $a^n \pm 1$ is a base $b$ palindrome contains an infinite arithmetical progression. When additionally all the prime factors of $b$ divide $a$, it is pointed out, without proof, that the set of such $n$ is finite and effectively computable. In this note, we give the details of the proof of this assertion and perform the computations for the case when $a = 10$ and $b = 2$.

Our result is the following:

**Theorem 1.1.** The only positive integers $n$ such that $10^n \pm 1$ are binary palindromes are 1, 2, i.e. $9 = 1001_{(2)}$ and $99 = 1100011_{(2)}$.

2. The proof

We assume first that $n > 1000$. We put $m$ for the number of binary digits of $N = 10^n \pm 1$. Since none of $10^n$ and $10^n \pm 1$ is a power of 2, it follows that

$$m = \lfloor (\log(10^n \pm 1))/\log 2 \rfloor + 1 = \lfloor n(\log 10)/\log 2 \rfloor + 1 = n + \lfloor n(\log 5)/\log 2 \rfloor + 1. $$

Put $\ell = \lfloor n(\log 5)/\log 2 \rfloor + 1$ for the number of binary digits of $5^\ell$. The last binary $n$ + 1 digits of $N$ are $100 \ldots 01$ with $n − 1$ zeros, or $011 \ldots 11$ with $n$ ones, depending on whether the sign is $+$ or $−$, respectively. Since $N$ is a binary palindrome, the first binary digits of $N$ must be $100 \ldots 01$ with $n − 1$ zeros and $11 \ldots 110$ with $n$ ones, when the sign is $+$ and respectively $−$. Since $2 < 5$, the first $n$ binary digits of $N$ are the same as the first $n$ binary digits of $5^\ell$. In fact, this is true for the first $\ell − 1 \geq n$ binary digits of $N$. Thus, in the $+$ case, we have

$$2^\ell < 5^n \leq 2^\ell + 2^{\ell−n} + 2^{\ell−n−1} + \cdots + 1 < 2^\ell + 2^{\ell−n+1},$$

while in the $−$ case

$$2^{\ell+1} > 5^n > 2^\ell + 2^{\ell−1} + \cdots + 2^{\ell−n+1} = 2^{\ell−n+1}(2^{n−1} + \cdots + 1) = 2^{\ell−n+1}(2^n − 1) = 2^{\ell+1} − 2^{\ell−n+1}.$$

Putting $k = \ell$ or $\ell + 1$ according to whether the sign is $+$ or $−$, we get that

$$|5^n 2^{−k} − 1| < 2^{\ell−n+1−k} \leq 2^{−n+1}.$$  

Put $A = n \log 5 − k \log 2$. The above inequality becomes $|e^A − 1| < 2^{−n+1}$. If $A > 0$, then

$$|A| = A < e^A − 1 < 2^{−n+1},$$

while if $A < 0$, then the inequality

$$|e^A − 1| = 1 − e^{−|A|} < 2^{−n+1}$$

leads to

$$|A| < e^{−|A|} − 1 < \frac{1}{1−2^{−n+1}} − 1 = \frac{2^{−n+1}}{1−2^{−n+1}} < 2^{−n+2}$$

because $n > 2$ (thus, $2^{−n+1} < 1/2$). In conclusion, the inequality

$$|A| < 2^{−n+2}$$

holds in both cases and it implies

$$(n − 2) \log 2 < − \log |A|. \quad (1)$$

We have now set ourselves up to apply a lower bound for a linear form in logarithms. There are many such in the literature. We have chosen to apply the following one which is Corollary 2 on page 228 in [5]. First, let us introduce
some notations. Let \( a_1 \) and \( a_2 \) be integers \( > 1 \), \( b_1 \) and \( b_2 \) be positive integers, \( A_1 \) and \( A_2 \) be positive real numbers such that \( A_i \geq \max\{a_i, e\} \) for \( i = 1, 2 \), and \( \Lambda = b_2 \log a_2 - b_1 \log a_1 \). Let

\[
b' = \frac{b_1}{\log A_2} + \frac{b_2}{\log A_1}.
\]

With these notations, the above mentioned result from [5] asserts that

\[
-\log |\Lambda| \leq 24.34 \left( \max\{\log b' + 0.14, 21\} \right)^2 \log A_1 \log A_2.
\]

For us, we can take \( a_1 = 2, a_2 = 5, b_1 = k, b_2 = n, A_1 = e \) and \( A_2 = 5 \). Thus,

\[
b' = \frac{k}{\log 5} + n \leq \frac{\ell + 1}{\log 5} + n \leq \frac{1}{\log 5} \left( \frac{n \log 5}{\log 2} + 2 \right) + n \leq n + \frac{n}{\log 2} + \frac{2}{\log 5} \leq 3n + 2. \tag{3}
\]

Combining estimates (1) with (2) and (3), we get

\[
(n - 2) \log 2 < 24.34 \max\{\log(3n + 2) + 0.14, 21\}^2 \log 5. \tag{4}
\]

The last inequality above implies that \( n \leq 5500 \). A few seconds of computation with Mathematica revealed that in the range \( n \leq 5500 \), there are no other binary palindromes among \( 10^n \pm 1 \) except for the two shown in the statement of the theorem, which completes the proof.

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**References**


