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Dynamical Systems

No finite invariant density for Misiurewicz exponential maps ☆

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Abstract

For exponential mappings such that the orbit of the only singular value 0 is bounded, it is shown that no integrable density invariant under the dynamics exists on $\mathbb{C}$. ***To cite this article: J. Kotus, G. Świątek, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Il n'existe aucune densité intégrable pour des applications exponentielles de Misiurewic. Pour les applications exponentielles de $\mathbb{C}$ dont l'orbite de la valeur singulière 0 est bornée, on montre qu'il n'existe aucune densité intégrable et invariante sous la dynamique. ***Pour citer cet article : J. Kotus, G. Świątek, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

1. Introduction

We consider one parameter family of exponential functions $f_\Lambda(z) = \Lambda e^z$, $z \in \mathbb{C}$, $\Lambda \in \mathbb{C}^*$. These maps have only one finite singular value 0 whose forward trajectory determines the dynamics on $\mathbb{C}$. From now we assume that the orbit of the asymptotic value 0 is bounded and the Julia set $J(f_\Lambda) = \mathbb{C}$. Thus f satisfies so called Misiurewicz condition i.e. the post-singular set $P(f) := \overline{\bigcup_{n=0}^{\infty} f_\Lambda^n(0)}$ is bounded and $P(f) \cap \mathrm{Crit}(f) = \emptyset$. It follows from [4, Th. 1] that $P(f)$ is hyperbolic. The problem of existence of probabilistic invariant measure absolutely continuous with respect to the Lebesgue measure (abbr. *pacim*) for transcendental meromorphic functions satisfying Misiurewicz condition was discussed in [6]. However, this result cannot be applied to entire functions. The main result of this Note is the following theorem:

Theorem 1. *Let $f(z) = \Lambda \exp(z)$ with $\Lambda \in \mathbb{C} \setminus \{0\}$ chosen so that the Julia set is the entire sphere and the orbit of* 0 *under f is bounded. Then f admits no probabilistic invariant measure absolutely continuous with respect to the Lebesgue measure.*

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However these maps have σ-finite invariant measure absolutely continuous with respect to the Lebesgue measure (see [5]). A result similar to Theorem 1 has been mentioned to us by other authors, [2].

The proof will proceed by contradiction, so we suppose that such a measure exists and call it μ, while reserving λ for the Lebesgue measure of the plane. It follows from [3] that the set of points escaping to ∞ has zero Lebesgue's measure for every map in our family. It is not difficult to prove that for these functions the union $P(f) \cup \{\infty\}$ is not a metric attractor in sense of Milnor with respect to the measure λ on $\mathbb{C}$. The results of [1] implies that f_Λ is ergodic with respect to λ. Thus

Fact 1. *The measure μ is ergodic.*

2. Proof

For a positive integer n write $A_n := \{z\colon\ |\Lambda|e^n < |z| \leqslant |\Lambda|e^{n+1},\ \arg z \neq \arg \Lambda\}$. A *fundamental rectangle* will refer to any set in the form $\{x + 2\pi i y\colon\ k < x < k+1,\ l < y < l+1\}$ for integers k, l. Thus, any fundamental rectangle is mapped with bounded distortion and onto some A_n.

Lemma 1. *For all $n \in \mathbb{Z}_+$, $\inf\mathrm{ess}\{\frac{d\mu}{d\lambda}(z)\colon\ z \in A_n\} > 0$.*

Proof. By [4, Th. 1], the post-singular set $P(f)$ has area 0, so it cannot be the support of μ. Additionally, the image of every open set covers A_n after finitely many iterations, so it suffices to have the $\frac{d\mu}{d\lambda}$ essentially bounded away from 0 on any open set. Hence, Lemma 1 follows from the following fact: □

Lemma 2. *Suppose that F is a meromorphic function whose Julia set is the entire sphere, and ν a probability invariant and ergodic measure absolutely continuous with respect to λ and such that the ν-measure of the closure of the post-singular set of F is less than 1. Then, there is an open set U such that*

$$\inf\mathrm{ess}\left\{\frac{d\nu}{d\lambda}(z)\colon\ z \in U\right\} > 0.$$

Proof. Fix U to be a disk in a positive distance from the post-critical set of F and such that $\eta := \nu(U)$ is positive. Denote $\rho(z) := \frac{d\nu}{d\lambda}$. Pick $\epsilon > 0$. In the argument to follow it is important distinguish between parameters that do or do not depend on ϵ.

A variant of Luzin's Theorem. For every $\epsilon > 0$, we can find a continuous function with compact support $\rho_\epsilon : \mathbb{C} \to [0, +\infty)$ such that

$$\int_{\mathbb{C}} \bigl(\rho_\epsilon(w) - \rho(w)\bigr)_+ \, d\lambda(w) < \epsilon \tag{1}$$

where the plus subscript denote the positive part,

$$\int_{\mathbb{C}} \min\bigl(\rho_\epsilon(z), \rho(z)\bigr) \, d\lambda(z) \geqslant 1 - \eta/10. \tag{2}$$

This statement follows from introductory measure theory.

Proof of Lemma 2 continued. Now for any k consider the set Ω_k of connected components of $F^{-k}(U)$ which intersect the support of ρ_ϵ. If $V \in \Omega_k$, then F^k maps V onto U univalently and with distortion bounded depending solely on U. Denote $d_k = \sup\{\mathrm{diam}\, V\colon\ V \in \Omega_k\}$. Since the Julia set is the whole sphere, $\lim_{k\to\infty} d_k = 0$. Let G_k denote the set of inverse branches of F^k defined on U. For z in U $\rho_{\epsilon,k}(z) = \sum_{g\in G_k} \inf\{\rho_\epsilon(w)\colon\ w = g(z), z \in U\}|g'(z)|^2$. For any g, the ratio of the values of each summand at two points z_1, z_2 is equal to the ratio of $|g'|^2$ at these points, hence bounded above by some $Q_0 \geqslant 1$ which depends solely on the distortion of inverse branches and therefore only on U. Consequently,

$$\frac{\rho_{\epsilon,k}(z_1)}{\rho_{\epsilon,k}(z_2)} \leqslant Q_0 \tag{3}$$

for every $z_1, z_2 \in U$. Consider a similarly constructed $\tilde{\rho}_\epsilon(z) = \sum_{g \in G_k} \rho_\epsilon(g(z))|g'(z)|^2$. By the change of variable formula

$$\int_U \tilde{\rho}_\epsilon(z)\,\mathrm{d}\lambda(z) = \int_{F^{-k}(U)} \rho_\epsilon(w)\,\mathrm{d}\lambda(w) \geqslant \int_{F^{-k}(U)} \min\bigl(\rho(w), \rho_\epsilon(w)\bigr)\,\mathrm{d}\lambda(w)$$
$$= \int_{\mathbb{C}} \min\bigl(\rho(w), \rho_\epsilon(w)\bigr)\,\mathrm{d}\lambda(w) - \int_{F^{-k}(U)^c} \min\bigl(\rho(w), \rho_\epsilon(w)\bigr)\,\mathrm{d}\lambda(w)$$
$$\geqslant 1 - \eta/10 - \nu\bigl(F^{-k}(U)^c\bigr) = 1 - \eta/10 - (1-\eta) = \frac{9}{10}\eta \tag{4}$$

where we have also used condition (2). Clearly, $\rho_{\epsilon,k} \leqslant \tilde{\rho}_\epsilon$. Let δ_ϵ denote the modulus of continuity of ρ_ϵ. Then

$$\int_U \bigl(\tilde{\rho}_\epsilon(z) - \rho_{\epsilon,k}(z)\bigr)\,\mathrm{d}\lambda(z) \leqslant \delta_\epsilon(d_k) \int_U \sum_{g \in G'_k} \bigl|g'(z)\bigr|^2\,\mathrm{d}\lambda(z).$$

Here G'_k denoted the set of only those inverse branches which map onto some $V \in \Omega_k$. By bounded distortion, if g maps on V, then for any $z \in U$, $|g'(z)|^2 \leqslant Q_0 \frac{\lambda(V)}{\lambda(U)}$. Hence, we can further estimate

$$\int_U \bigl(\tilde{\rho}_\epsilon(z) - \rho_{\epsilon,k}(z)\bigr)\,\mathrm{d}\lambda(z) \leqslant \delta_\epsilon(d_k)\lambda(U)^{-1} \sum_{V \in \Omega_k} \lambda(V).$$

Since all $V \in \Omega_k$ must touch the compact support of ρ_ϵ and their diameters tend uniformly to 0 with k, their joint area remains bounded depending solely on U, ϵ. Since also d_k tend to 0 with k, for all $k \geqslant k(\epsilon)$,

$$\int_U \bigl(\tilde{\rho}_\epsilon(z) - \rho_{\epsilon,k}(z)\bigr)\,\mathrm{d}\lambda(z) \leqslant \frac{2}{5}\eta.$$

Taking into account estimate (4), for $k \geqslant k(\epsilon)$, $\int_U \rho_{\epsilon,k}(z)\,\mathrm{d}\lambda(z) \geqslant \eta/2$. Based on estimate (3), we conclude that for all $k \geqslant k(\epsilon)$,

$$\rho_{\epsilon,k}(z) \geqslant Q_1 > 0 \tag{5}$$

for all $z \in U$ and Q_1 which only depends on U. Next, we estimate

$$\int_U \bigl(\rho_{\epsilon,k}(z) - \rho(z)\bigr)_+\,\mathrm{d}\lambda(z) \leqslant \int_U \bigl(\tilde{\rho}_\epsilon(z) - \rho(z)\bigr)_+\,\mathrm{d}\lambda(z) = \int_{\mathbb{C}} \bigl(\rho_\epsilon(w) - \rho(w)\bigr)_+\,\mathrm{d}\lambda(w) < \epsilon$$

where we used a change of variables formula and condition (1). For every $\epsilon > 0$ and $k \geqslant k(\epsilon)$, we conclude from this and estimate (5) that $\rho(z) < \frac{Q_1}{2}$ on a set λ-measure less than $\frac{2\epsilon}{Q_1}$. Since ϵ can be made arbitrarily small while Q_1 is fixed, then $\rho(z) \geqslant \frac{Q_1}{2}$ on a set of full λ-measure in U. □

2.1. Return times

Introduce the following function $g : \mathbb{R} \to \mathbb{R}$: $g(x) = |\Lambda|\sqrt{\mathrm{e}^x}$.

Lemma 3. *There exists N_0 such that for all $n \geqslant N_0$, there exist sets $W_+, W_- \subset A_n$ which consist of fundamental rectangles each of which is mapped by f onto some $A_m \subset \{z \in \mathbb{C}\colon |z| \geqslant g(|\Lambda|\mathrm{e}^n)\}$ in the case of W_+, $A_m \subset \{z \in \mathbb{C}\colon |z| \leqslant g(-|\Lambda|\mathrm{e}^n)\}$ for W_- and such that*

$$\lambda(W_\pm) > \frac{1}{4}\lambda(A_n).$$

Proof. For an annulus centered at 0 with inner radius r, $1/3$ of its area belongs to the half-plane $\Re z > r/2$ and another $1/3$ to $\Re z < -r/2$. For A_n with n large enough, almost the entire area, certainly more than $1/4$ of the area of the whole annulus, of $A_n \cap \{z\colon \Re z > |\Lambda| \exp n\}$ can be filled with fundamental rectangles. This defines W_+. The set W_- is constructed in the same way. □

The following lemma generalizes Lemma 3:

Lemma 4. *There are constants N_1 and $K_0 > 1$ such that for all $n \geqslant N_1$ and any integer $p \geqslant 1$, there is a set $W_p \subset A_n$ such that*:

- *W_p is the union of sets each of which is mapped by f^{p-1} univalently onto a fundamental rectangle,*
- *for every $z \in W_p$ and $0 < j < p$, $f^j(z) \in A_m$ with $m \geqslant n$, while $f^p(z) \in A_m$ with $m \geqslant g^p(|\Lambda|\mathrm{e}^n)$,*
- *$\lambda(W_p) \geqslant K_0^{-p}$.*

Proposition 1. *There exist constants N_2 and $K_0, K_1 > 1$ such that for each $n \geqslant N_2$ and $p \geqslant 1$, A_n contains a subset V_p, such that V_p are pairwise disjoint for different p and for every $z \in V_p$, $|f^i(z)| \geqslant |\Lambda|\mathrm{e}^n$ for $i = 0, \ldots, p$ while $|f^{p+1}(z)| \leqslant g(-g^p(|\Lambda|\mathrm{e}^n))$. Additionally, for each p, $\lambda(V_p) \geqslant K_1^{-1} K_0^{-p} \lambda(A_n)$.*

Proof of the proposition. We choose N_2 at least equal to N_1 from Lemma 4, such that $g(|\Lambda|\mathrm{e}^n) \geqslant |\Lambda|\mathrm{e}^n$ if $n \geqslant N_2$ and so big that the orbit 0 fits inside $D(0, |\Lambda|\mathrm{e}^{N_2-1})$ and at least 1. By the last choice, the pairwise disjointness of sets V_p will follow automatically from the conditions on orbits from V_p. Consider first the set W_p obtained from Lemma 4. It consists of sets U_j which are univalent preimages of fundamental rectangles, each of which is mapped with bounded distortion onto $A_m \subset \{z \in \mathbb{C}\colon |z| \geqslant g^p(|\Lambda|\mathrm{e}^n)\}$. Thus, a portion of U_j of area at least $K_1^{-1}\lambda(U_j)$ with K_1 a constant, is occupied by the preimage by f^p of the set W_- from Lemma 3. It is immediate that every z from this preimage satisfies the demands of Proposition 1. V_p is the union of such preimages for all U_j and hence its measure is bounded below as claimed in the proposition. □

Proof of Theorem 1.

Lemma 5. *For all $x \geqslant N_3$ for some N_3 and every $\gamma > 0$, $\lim_{p\to\infty} g^p(x)\gamma^{-p} = +\infty$.*

Consider a slit annulus A_n for n at least equal to the constant N_2 of Proposition 1 and $|\Lambda|\mathrm{e}^n \geqslant N_3$ of Lemma 5. Let $\tau(z)$ for $z \in A_n$ be the first return time to A_n. Note that μ-almost every point returns since open sets return and μ is ergodic. Clearly τ is μ-integrable, but then also λ-integrable in view of Lemma 1. Similarly, λ-almost every point returns. If $z \in D(0, r)$ then it takes at least $k \geqslant K_2 \log r^{-1}$ for $f^k(z)$ to get in the distance at least 1 unit away from the orbit of 0. K_2 is a positive constant which depends on the maximum modulus of the derivative of f on some compact set. It follows that on each set V_p from Proposition 1, the return time is at least $K_2(\log|\Lambda| + g^p(|\Lambda|\mathrm{e}^n))$. Since the measure of V_p is only exponentially small with p, by Lemma 5, the return time is not λ-integrable which gives us the final contradiction. □

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