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Partial Differential Equations

Gradient and Hölder estimates for positive solutions of Pucci type equations

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Abstract

We present some estimates for positive viscosity solutions of a class of fully non-linear elliptic equations including the extremal Pucci equations, generalizing some results for linear equations recently established by Y.Y. Li and L. Nirenberg. ***To cite this article: I. Capuzzo Dolcetta, A. Vitolo, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Estimations Hölder et du gradient pour les solutions non négatives des équations de Pucci. Le but de cette Note est de donner des estimations pour les solutions de viscosité non négatives d'une classe d'équations complètement non linéaires comprenante les équations extrémales de Pucci, en généralisant ainsi des résultats récents dues à Y.Y. Li et L. Nirenberg. ***Pour citer cet article : I. Capuzzo Dolcetta, A. Vitolo, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Version française abrégée

Le but de cette Note est d'établir des estimations du gradient et de type Hölder pour les solutions de viscosité non négatives d'une classe d'équations elliptiques complètement non linéaires de la forme $F(D^2u) = f$ dans une boule $B_R(x_0)$ de $\mathbb{R}^n$. Les recherches qu'on présente ici sont motivés par un article récent de Y.Y. Li et L. Nirenberg. La classe des équations considérées inclut en particulier les équations extremales de Pucci

$$\mathrm{P}^+_{\lambda,\Lambda}(D^2u) = \sup_{A\in \mathrm{A}_{\lambda,\Lambda}} \mathrm{Tr}(AD^2u) = f \quad \text{et} \quad \mathrm{P}^-_{\lambda,\Lambda}(D^2u) = \inf_{A\in \mathrm{A}_{\lambda,\Lambda}} \mathrm{Tr}(AD^2u) = f.$$

Nos résultats peuvent donc être regardés comme des généralisations de ceux dans [8]. Un premier résultat est la validité des inégalités de Glaeser

$$\bigl|Du(x)\bigr| \leqslant C\sqrt{u(x_0)M} \quad \text{si } 2|x| \leqslant \sqrt{\frac{u(x_0)}{M}} \leqslant R; \quad \bigl|Du(x)\bigr| \leqslant C\left(\frac{u(x_0)}{R} + MR\right) \quad \text{si } 2|x| \leqslant R \leqslant \sqrt{\frac{u(x_0)}{M}},$$

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où $M = \sup_{B_R(x_0)} |f|$, dans tout point x de différentiabilité de la solution $u \geqslant 0$ de $F(D^2u) = f$, pourvu que F satisfasse une certaine propriété d'invariance par réflexion, voir Lemma 1.1 et Proposition 1.1.

En ce qui concerne les estimations Hölderiennes, on démontre que toute solution de viscosité non négative de l'équation $F(D^2u) = f$ avec $\|f\|_{L^n(B_R(x_0))} \leqslant M$ vérifie

$$\sup_{\substack{x,y \in B_R(x_0)\\ x \neq y}} \frac{|u(x) - u(y)|}{|x - y|^\alpha} \leqslant C\left(\frac{u(x_0)}{R^\alpha} + MR^{1-\alpha}\right),$$

avec $\alpha \in (0, 1)$ et $C > 0$ dépendantes de n, λ, Λ.

Les démonstrations reposent sur des techniques de comparaison, voir le Lemma 1.1, et sur la version, voir [3], de l'inégalité de Harnack

$$\sup_{B_{\frac{3}{4}r}(x)} u \leqslant C_2\left(\inf_{B_{\frac{3}{4}r}(x)} u + r\|f\|_{L^n(B_r(x))}\right)$$

pour les solutions de viscosité non négatives de $F(D^2u) = f$.

1. Introduction and results

A well-known estimate for classical solutions $u \in C^2(B_d) \cap C^0(\bar{B}_d)$ of the Poisson's equation $\Delta u = f$ in the ball $B_d = \{x =\in \mathbb{R}^n \mid |x| < d\}$ is

$$\left|Du(0)\right| \leqslant \frac{n\sqrt{2}}{d} \sup_{B_d} |u| + \frac{d}{2\sqrt{2}} \sup_{B_d} |f|, \tag{1}$$

where Du is the gradient of u, see [6]. In a recent paper, combining estimates of this kind with the Harnack inequality, Y.Y. Li and L. Nirenberg [8] obtained for non-negative solutions $u \in C^2(B_R(x_0))$ of the Poisson's equation an extension of Glaeser's one-dimensional inequality. Their result is that, for $M = \sup_{B_R(x_0)} |f|$, the following estimates hold:

$$\left|Du(x)\right| \leqslant C\sqrt{u(x_0)M} \quad \text{if } 2|x| \leqslant \sqrt{\frac{u(x_0)}{M}} \leqslant R; \qquad \left|Du(x)\right| \leqslant C\left(\frac{u(x_0)}{R} + MR\right) \quad \text{if } 2|x| \leqslant R \leqslant \sqrt{\frac{u(x_0)}{M}},$$

for some positive constant $C = C(n)$.

Our aim here is to extend the validity of the above inequalities, as well as of (1), to non-negative generalized solutions in the viscosity sense, see [4], of the fully non-linear equation

$$F\big(D^2u(x)\big) = f(x) \tag{2}$$

where D^2u denotes the Hessian matrix of u. The leading requirements we make on F are uniform ellipticity and a reflection invariance property. For clarity of exposition we will also assume $F(0) = 0$. By uniform ellipticity, we mean that, for some constants $0 < \lambda \leqslant \Lambda$,

$$\lambda \operatorname{Tr}(Y) \leqslant F(X + Y) - F(X) \leqslant \Lambda \operatorname{Tr}(Y)$$

for $X, Y \in S^n$ with $Y \geqslant 0$, where S^n and Tr denote, respectively, the space of real symmetric $n \times n$ matrices endowed with the partial ordering induced by non-negative definiteness and the trace of such a matrix. Note that (2) reduces to the Poisson's equation for $F(X) = \operatorname{Tr}(X)$.

The mapping F is reflection invariant with respect to a hyperspace H if

$$F(X) = F(RXR) \quad \text{for all } X \in S^n$$

where $R \in S^n$ be the reflection matrix with respect to H. Inequality (1) can be extended to our setting as follows:

Lemma 1.1. *Let $u \in C^0(B_d)$ be a viscosity solution of* (2)*. Suppose that F is uniformly elliptic and is invariant by reflection with respect to n orthogonal hyperspaces $H_1, \dots, H_n$. If u is differentiable at $x = 0$, then*

$$\max_{i=1,\dots,n} \left|u_{x_i}(0)\right| \leqslant \frac{n}{d}\sqrt{\frac{\lambda + \Lambda}{\lambda}} \sup_{B_d} |u| + \frac{d}{2} \frac{1}{\sqrt{\lambda(\lambda + \Lambda)}} \sup_{B_d} |f|. \tag{3}$$

If F depends on matrix X only through its eigenvalues, then any unit vector defines an invariant by reflection hyperplane for F. In this case, $\max_i |u_{x_i}(0)| = |Du(0)|$ in Eq. (3) while in general $\max_i |u_{x_i}(0)| \geqslant \frac{|Du(0)|}{\sqrt{n}}$. Combining the result of Lemma 1.1 with the Harnack inequality for non-negative viscosity solutions of (2), see [3], we deduce:

Proposition 1.1. *Let F be as in Lemma 1.1 and assume moreover that $F(0) = 0$. If u is a non-negative viscosity solution of Eq. (2) in the ball $B_R(x_0)$ with $f \in C^0(B_R(x_0))$ and $|f| \leqslant M$ in $B_R(x_0)$, then the estimates*

$$\left|Du(x)\right| \leqslant C\sqrt{u(x_0)M} \quad \text{if } 2|x| \leqslant \sqrt{\frac{u(x_0)}{M}} \leqslant R, \tag{4}$$

$$\left|Du(x)\right| \leqslant C\left(\frac{u(x_0)}{R} + MR\right) \quad \text{if } 2|x| \leqslant R \leqslant \sqrt{\frac{u(x_0)}{M}} \tag{5}$$

hold at any differentiability point $x \in B_{R/2}(x_0)$, for some constant C depending on n, λ, Λ.

A result due to Ishii–Lions, see [7] Theorem 7.2, applies in our setting guaranteeing the Lipschitz continuity of viscosity solutions of (2); hence, by the Rademacher Theorem, the estimates (4), (5) hold in fact almost everywhere in $B_{R/2}(x_0)$. Concerning Hölder estimates, we have the following extension of a linear result in [8]:

Proposition 1.2. *Assume that F is uniformly elliptic with $F(0) = 0$. If $u \in C^0(B_R(x_0))$ is a non-negative viscosity solution of (2) and $\|f\|_{L^n(B_R(x_0))} \leqslant M$, then*

$$\sup_{\substack{x,y\in B_R(x_0)\\ x\neq y}} \frac{|u(x) - u(y)|}{|x - y|^\alpha} \leqslant C\left(\frac{u(x_0)}{R^\alpha} + MR^{1-\alpha}\right), \tag{6}$$

for some positive constants $\alpha \in (0, 1)$ and C depending on n, λ, Λ.

The proofs of Lemma 1.1 and of Propositions 1.1, 1.2 are outlined in Section 2.

We conclude this section with a few examples of equations fitting in our framework. We denote by $\mathrm{A}_{\lambda,\Lambda}$ denotes the closed convex set in S^n consisting of positive definite matrices whose eigenvalues belong to the interval $[\lambda, \Lambda]$, $\lambda > 0$. The Pucci extremal equations

$$\mathrm{P}^+_{\lambda,\Lambda}(D^2u) = \sup_{A\in \mathrm{A}_{\lambda,\Lambda}} \mathrm{Tr}(AD^2u) = f \quad \text{and} \quad \mathrm{P}^-_{\lambda,\Lambda}(D^2u) = \inf_{A\in \mathrm{A}_{\lambda,\Lambda}} \mathrm{Tr}(AD^2u) = f$$

are basic models of fully non-linear equations satisfying our leading assumptions. Using the alternative representation, see [3],

$$\mathrm{P}^+_{\lambda,\Lambda}(X) = \Lambda\,\mathrm{Tr}(X^+) - \lambda\,\mathrm{Tr}(X^-), \qquad \mathrm{P}^-_{\lambda,\Lambda}(X) = \lambda\,\mathrm{Tr}(X^+) - \Lambda\,\mathrm{Tr}(X^-),$$

where $X^\pm$ are non-negative definite matrices such that $X = X^+ - X^-$, $X^+X^- = 0$, it is easy to check that $\mathrm{P}^\pm_{\lambda,\Lambda}$ are invariant by reflection with respect to any orthogonal set of hyperspaces. In the sequel we will make use of the fact that Pucci operators are positively homogeneous of degree 1 and also of the inequalities

$$\mathrm{P}^\pm_{\lambda,\Lambda}(X) + \mathrm{P}^-_{\lambda,\Lambda}(Y) \leqslant \mathrm{P}^\pm_{\lambda,\Lambda}(X + Y) \leqslant \mathrm{P}^+_{\lambda,\Lambda}(X) + \mathrm{P}^\pm_{\lambda,\Lambda}(Y).$$

Note that the above inequalities imply in particular that $\mathrm{P}^\pm_{\lambda,\Lambda}$ are uniformly elliptic with ellipticity constants λ and Λ. More generally, reflection invariance holds for any F depending only on the eigenvalues of X, since such an F is not changed by the action O^TXO of orthogonal matrices O. Linear equations of the form $\mathrm{Tr}(AD^2u) = f$ with $A \in \mathrm{A}_{\lambda,\Lambda}$ are of course a special subcase.

The Bellman equations in stochastic optimal control,

$$\inf_{k\in K} \mathrm{Tr}(A_kD^2u) = f$$

with $A_k \in \mathrm{A}_{\lambda,\Lambda}$, $k \in K$, fit in our framework provided that A_k commutes with A_j for each $k, j \in K$. If so, then one can find $\xi_1, \dots, \xi_n$ in $\mathbb{R}^n$ forming a common orthonormal set of eigenvectors for all matrices A_k; therefore $\inf_{k\in K} \mathrm{Tr}(A_kX)$ is invariant by reflection with respect to the hyperspaces of equations $\xi_1 \cdot x = 0, \dots, \xi_n \cdot x = 0$. Similar arguments can be applied to discuss the more general case of Isaacs equations $\sup_{j\in K} \inf_{k\in K} \mathrm{Tr}(A_{k,j}D^2u) = f$.

2. Proofs

If F is invariant by reflection with respect to a set of orthogonal hyperspaces $H_1, \dots, H_n$, then the reflection matrices are of the form $R_i = O^T R_i^* O$ where O is an orthogonal matrix and R_i^*, $i = 1, \dots, n$, are the reflection matrices with respect to the hyperspaces $x_1 = 0, \dots, x_n = 0$. This implies that the mapping $G(X) = F(O^T X O)$ is invariant by reflection with respect to $x_1 = 0, \dots, x_n = 0$, as it follows by the identities $G(R_i^* X R_i^*) = F(O^T R_i^* X R_i^* O) = F(R_i O^T X O R_i) = F(O^T X O) = G(X)$.

Clearly, if F is uniformly elliptic, the same is true for G. Observe also that if u is a smooth solution of (2), then $v(y) = u(O^T x)$ satisfies $G(D^2 v(y)) = g(y)$ where $g(y) = f(O^T y)$, since $D^2 u(x) = O^T D^2 v(y) O$. It is not difficult to check that the same holds true for viscosity solutions of (2).

Proof of Lemma 1.1. In view of the previous discussion, we may assume that F is invariant by reflection with respect to the cohordinate hyperspaces of equations $x_1 = 0, \dots, x_n = 0$. Set $\bar{d} = \frac{d}{\sqrt{\Lambda+\lambda}}$ so that the open cylinder

$$K = K_{\bar{d}} = \left\{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid |x'| < \bar{d}\sqrt{\Lambda},\ |x_n| < \bar{d}\sqrt{\lambda}\right\}$$

is contained in B_d. In order to prove the estimate (3) we can suppose that u is continuous up to the boundary, otherwise we carry out the calculations below in the smaller cylinder $K_{(1-\varepsilon)d}$ and then let $\varepsilon \to 0^+$. Denote by K' the upper half-cylinder $K' = \{x \in K\colon\ x_n > 0\}$ and by u^* the function $u^*(x', x_n) = u(x^*)$, where $x^* = (x', -x_n)$. If u is a smooth solution of (2), then $D^2 u^*(x) = R_n^* D^2 u(x^*) R_n^*$. Hence, the reflection invariance of F with respect $x_n = 0$ yields

$$F(D^2 u^*)(x) = f^*(x), \qquad x \in K', \tag{7}$$

where $f^*(x) = f(x^*)$. It is not hard to check that the same conclusion holds for viscosity solutions. Setting $M = \sup_K |f|$, from Eqs. (2) and (7) we deduce that

$$-M \leqslant F(D^2 u) \leqslant M, \qquad -M \leqslant F(D^2 u^*) \leqslant M$$

in the viscosity sense. Consider now the function

$$\tilde{u}(x) = \frac{1}{2}\big(u(x) - u^*(x)\big) = \frac{1}{2}\big(u(x', x_n) - u(x', -x_n)\big), \quad x \in K'.$$

A slight modification of Theorem 5.3 in [3] allows one to conclude that

$$-M \leqslant \mathrm{P}^+_{\lambda,\Lambda}(D^2\tilde{u}), \quad \mathrm{P}^-_{\lambda,\Lambda}(D^2\tilde{u}) \leqslant M \tag{8}$$

in K', in the viscosity sense. Define next, for $x \in K'$ the smooth comparison function

$$\Phi(x) = \frac{N}{\bar{d}^2}\left[\frac{|x'|^2}{\Lambda} + \frac{x_n}{\sqrt{\lambda}}\left(n\bar{d} - (n-1)\frac{x_n}{\sqrt{\lambda}}\right)\right] + \frac{M}{2}\frac{x_n}{\sqrt{\lambda}}\left(\bar{d} - \frac{x_n}{\sqrt{\lambda}}\right)$$

where $N = \sup_K |u|$. A direct computation shows that

$$\mathrm{P}^+_{\lambda,\Lambda}(D^2\Phi) = \Lambda\frac{2N}{\Lambda\bar{d}^2}(n-1) - \lambda\left(\frac{2N}{\lambda\bar{d}^2}(n-1) + \frac{M}{\lambda}\right) = -M.$$

Using this fact, the inequalities (8) and Lemma 2.12 in [3], we deduce, since Φ is smooth, that

$$\mathrm{P}^+_{\lambda,\Lambda}\big(D^2(\tilde{u} - \Phi)\big) \geqslant 0 \geqslant \mathrm{P}^-_{\lambda,\Lambda}\big(D^2(\tilde{u} + \Phi)\big) \quad \text{in } K'.$$

Since $\tilde{u} - \Phi \leqslant 0 \leqslant \tilde{u} + \Phi$ on $\partial K'$, then by the weak Maximum Principle, $\tilde{u} - \Phi \leqslant 0 \leqslant \tilde{u} + \Phi$ for all $x \in K'$. Take now $x' = 0$ in the above and divide by $x_n > 0$ to get

$$\frac{|u(0, x_n) - u(0, -x_n)|}{2x_n} \leqslant \frac{N}{\bar{d}^2}\left(n\bar{d} - (n-1)\frac{x_n}{\sqrt{\lambda}}\right) + \frac{M}{2\sqrt{\lambda}}\left(\bar{d} - \frac{x_n}{\sqrt{\lambda}}\right).$$

Letting $x_n \to 0^+$ we obtain

$$\left|u_{x_n}(0)\right| \leqslant \frac{n}{\bar{d}}\sup_K |u| + \frac{\bar{d}}{2\sqrt{\lambda}}\sup_K |f|. \qquad \square$$

We need next the Harnack inequality, see Theorem 4.3 of [3], for continuous viscosity solutions $u \geqslant 0$ of Eq. (2) in $B_r(x)$:

$$\sup_{B_{\frac{3}{4}r}(x)} u \leqslant C_2\Big(\inf_{B_{\frac{3}{4}r}(x)} u + r\|f\|_{L^n(B_r(x))}\Big). \tag{9}$$

Proof of Proposition 1.1. By the Lipschitz regularity Theorem 7.2 of [7], any continuous viscosity solution u of (2) is differentiable a.e. in $B_R(x_0)$. Let $0 < r < R$ and $x \in B_{\frac{r}{2}}(x_0)$; take then $d = \frac{1}{4}r$, so that $B_d(x) \subset B_{\frac{3}{4}r}(x_0)$. By translational invariance we can use inequality (3) and then the Harnack inequality to get

$$\big|Du(x)\big| \leqslant C_1\left(\frac{\sup_{B_d(x)} u}{r} + Mr\right) \leqslant C_1\left(C_2\frac{u(x_0) + Mr^2}{r} + Mr\right).$$

It easily follows that $|Du(x)| \leqslant C(\frac{u(x_0)}{r} + Mr)$ at those $x \in B_R(x_0)$ where u is differentiable.

At this point, the inequalities (4) and (5) are derived arguing as in the proof of Proposition 2 of [8]. □

Proof of Proposition 1.2. If $u \in C^0(B_R(x_0))$ is a viscosity solution of Eq. (2), then the uniform ellipticity of F yields the C^α-estimate

$$R^\alpha \sup_{\substack{x,y\in B_R(x_0)\\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\alpha} \leqslant C\Big(\sup_{B_{\frac{3}{4}R}(x_0)} |u| + R\|f\|_{L^n(B_R(x_0))}\Big) \tag{10}$$

which is in fact a rescaled version of the Hölder estimate stated in [3], Proposition 4.10(2). Combined with the Harnack inequality (9), estimate (10) gives the result. □

3. Further remarks

Assuming, instead of our reflection invariance condition, that the following condition holds:

there exist positive constants $\bar\alpha < 1$ and $\bar C$ such that all solutions w of the homogeneous equation $F(D^2w) = 0$ in B_d, satisfy the a priori estimate

$$\|w\|_{C^{1,\bar\alpha}(B_{\frac{r}{2}}(x))} \leqslant \bar C r^{-(1+\bar\alpha)}\|w\|_{L^\infty(B_r(x))}, \quad 0 < r \leqslant d \tag{11}$$

at $x = 0$, then a qualitative version of inequality (3), namely

$$\big|Du(0)\big| \leqslant C_1\left(\frac{\sup_{B_d}|u|}{d} + d\sup_{B_d}|f|\right), \tag{12}$$

for some constant C_1, could be derived indeed from Theorem 2 in Caffarelli [2]. Note that (12) would be enough for the validity of Proposition 1.1. Note also that such an alternative approach based on assumption (11) allows one also to prove a $C^{1,\alpha}$ regularity result for viscosity solutions of Eq. (2):

Proposition 3.1. *Let F be a uniform elliptic operator such that $F(0) = 0$ and $|f| \leqslant M$ in $B_R(x_0)$. Assume moreover that* (11) *holds for all $x \in B_R(x_0)$. If u is a non-negative viscosity solution of* (2) *in $B_R(x_0)$, then $u \in C^{1,\alpha}(B_R(x_0))$ for every $0 < \alpha < \bar\alpha$ and the estimates* (4), (5) *hold in $B_{R/2}(x_0)$ with C depending on n, λ, Λ and $\bar\alpha$.*

The proof of Proposition 3.1 is similar to that of Proposition 1.1 but is based on Theorem 2 of [2], which yields $C^{1,\alpha}$- regularity and inequality (12), rather than on Theorem 7.2 of [7].

Note that condition (11) holds for convex equations thanks to the $C^{2,\alpha}$-estimates in [5], and also for the case where $F = \inf(F_1, F_2)$ with F_1 convex and F_2 concave, see Corollary 1.3(i) in [1].

We also point out that, if $u \in C^2(B_R)$ is solution of (2) with F smooth and $F(0) = 0$, then u solves a linear equation with continuous coefficients $a_{ij}(x) = \int_0^1 \frac{\partial F}{\partial X_{ij}}(tD^2u(x))\,\mathrm{d}t$. Then from Proposition 5 of [8] we get:

Proposition 3.2. *Let $u \in C^2(B_R(x_0))$ be a non-negative solution of* (2) *with F uniformly elliptic and smooth, such that $F(0) = 0$. If $|f(x)| \leqslant M$ in $B_R(x_0)$, then inequalities* (4) *and* (5) *hold in $B_{R/2}(x_0)$ with C depending on n, λ, Λ and the modulus of continuity of the a_{ij}'s defined above.*

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