



Optimal Control

Locally distributed desensitizing controls for the wave equation

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Abstract

We consider the wave equation with partially known initial data in a bounded domain. For this system, we construct locally distributed controls that desensitize a certain norm of the state. This result is new in space dimensions greater than one. The method of proof combines a judicious application of the Carleman estimate, and a localization technique. **To cite this article:** *L. Tebou, C. R. Acad. Sci. Paris, Ser. I 346 (2008).*

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Résumé

Contrôles insensibilisant localement distribués pour l'équation des ondes. Nous considérons dans un domaine borné une équation des ondes avec des données initiales incomplètes. Pour ce système, nous construisons des contrôles localement distribués qui insensibilisent une certaine norme de la solution du système. Ce résultat est nouveau pour les dimensions d'espace supérieures ou égales à deux. La méthode de démonstration allie une application judicieuse de l'inégalité de Carleman, et une technique de localisation. **Pour citer cet article :** *L. Tebou, C. R. Acad. Sci. Paris, Ser. I 346 (2008).*

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Soient Ω un ouvert connexe et borné de \mathbb{R}^d de classe C^2 , et $T > 0$. Soient \mathcal{O} , and ω deux ouverts non vides contenus dans Ω . Soit $\xi \in L^2(Q)$, où $Q = \Omega \times]0, T[$. On considère le système des ondes (1) où y^0, y^1 sont donnés dans des espaces de Hilbert adéquats, et \hat{y}^0, \hat{y}^1 sont des fonctions inconnues de norme unité dans les mêmes espaces. On se propose de trouver un contrôle v qui insensibilise (resp. ε -insensibilise) la fonctionnelle Φ définie par (2), c'est-à-dire que l'on va construire un contrôle v de sorte que l'on ait (3), respectivement (4).

La notion de contrôle insensibilisant une norme est bien connue pour ce qui est des équations paraboliques depuis le travail pionnier de Lions [10] suivi par de multiples travaux de différents autres auteurs [2–4,7,12,14,15]. Ce concept est mal connu pour les équations d'évolution du second ordre en temps ; à notre connaissance, seul l'article récent [5] en discute dans le cadre de l'équation des ondes unidimensionnelle. En effet, l'auteur de [5] montre entre autre que pour tout $(y^0, y^1) \in H_0^1(0, 1) \times L^2(0, 1)$, il existe un contrôle v qui ε -insensibilise Φ ; ce résultat est établi pour des ouverts non vides \mathcal{O} et ω arbitrairement choisis dans $(0, 1)$; il faut noter que les résultats démontrés dans tous

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les travaux antérieurs (à l'exception de [12]), portant par ailleurs sur les équations paraboliques, utilisent de façon essentielle la condition : $\mathcal{O} \cap \omega \neq \emptyset$. La méthode de démonstration dans [5] est basée sur le fait que l'équation homogène des ondes 1D est périodique en temps. Il est à noter que la périodicité en temps est perdue lorsque l'on passe en dimension d'espace supérieure à un, ou si, restant en dimension un, l'on considère des coefficients non constants ou même plus simplement l'équation des ondes avec un potentiel de coefficient constant. Ainsi l'approche adoptée dans [5] ne se généralise à aucune des situations nommées ci-avant ; une méthode nouvelle est nécessaire pour aborder ces problèmes. Ceci explique notre intérêt dans l'exploration de ce qui se passe dans le cas général ; bien que nous présentions notre méthode en utilisant l'équation des ondes, elle s'applique à une vaste classe d'équations hyperboliques [13].

Les résultats que nous avons obtenus sont les suivants :

Théorème 1. *Soient ω et \mathcal{O} deux ouverts quelconques contenus dans Ω satisfaisant : $\mathcal{O} \cap \omega \neq \emptyset$. Il existe T^* strictement positif tel que pour tous $T > T^*$, $y^0 \in H_0^1(\Omega)$, $y^1 \in L^2(\Omega)$, et pour tout ε strictement positif, il existe un contrôle $v \in L^2(0, T; L^2(\omega))$ qui ε -insensibilise la fonctionnelle Φ .*

Théorème 2. *Soient ω et \mathcal{O} deux voisinages de Γ_+ . Il existe T_0^* strictement positif tel que pour tous $T > T_0^*$, $y^0 \in L^2(\Omega)$, $y^1 \in H^{-1}(\Omega)$, et $\xi \in L^2(Q)$, il existe un contrôle $v \in [H^1(0, T; L^2(\omega))]'$ qui insensibilise la fonctionnelle Φ . De plus il existe une constante positive C telle que le contrôle v satisfait (5).*

1. Problem formulation and statements of main results

Let Ω be a connected and bounded open subset of \mathbb{R}^d , $d \geq 1$, with a boundary of class C^2 . Let $T > 0$. Let \mathcal{O} , and ω be two nonvoid open subsets in Ω . Let $\xi \in L^2(Q)$, where $Q = \Omega \times (0, T)$. We denote by ν the unit normal pointing into the exterior of Ω . We fix $x^0 \in \mathbb{R}^d$ and we set $m(x) = x - x^0$, and $\Gamma_+ = \{x \in \Gamma; m(x) \cdot \nu(x) > 0\}$, where $(u \cdot \nu = \sum_1^d u_i \nu_i$ for all $u, \nu \in \mathbb{R}^d$). Consider the wave system

$$\begin{cases} y_{tt} - \Delta y = \xi + v\chi_\omega & \text{in } Q, \\ y = 0 & \text{on } \Sigma = \partial\Omega \times (0, T), \\ y(0) = y^0 + \tau_0 \hat{y}^0; \quad y_t(0) = y^1 + \tau_1 \hat{y}^1 & \text{in } \Omega \end{cases} \quad (1)$$

where y^0, y^1 are given in appropriate Hilbert spaces, \hat{y}^0, \hat{y}^1 are unknown unit norm functions in the same Hilbert spaces, and τ_0 and τ_1 are small unknown real numbers. Our main purpose in this note is to find a control v that desensitizes (resp. ε -desensitizes) the functional Φ defined by

$$\Phi(y) = \frac{1}{2} \int_0^T \int_{\mathcal{O}} |y(x, t)|^2 dx dt, \quad (2)$$

that is to say, we are going to construct a control v satisfying for all \hat{y}^0 and \hat{y}^1 with unit norm in appropriate Hilbert spaces:

$$\left. \frac{\partial \Phi(y)}{\partial \tau_0} \right|_{\tau_0 = \tau_1 = 0} = 0 = \left. \frac{\partial \Phi(y)}{\partial \tau_1} \right|_{\tau_0 = \tau_1 = 0}, \quad (3)$$

respectively:

$$\forall \varepsilon > 0, \quad \left| \left. \frac{\partial \Phi(y)}{\partial \tau_0} \right|_{\tau_0 = \tau_1 = 0} \right| \leq \varepsilon, \quad \left| \left. \frac{\partial \Phi(y)}{\partial \tau_1} \right|_{\tau_0 = \tau_1 = 0} \right| \leq \varepsilon. \quad (4)$$

The notion of desensitizing control is well-known for parabolic equations since the pioneering work of Lions [10] subsequently followed by numerous works of different other authors [2–4, 7, 12, 14, 15]. This concept is ill-known for second order evolution equations; to our knowledge, there is only one recent paper [5] in the literature addressing the problem of desensitizing controls for the one-dimensional wave equation. In fact the author of [5] shows, among other things, that for all $(y^0, y^1) \in H_0^1(0, 1) \times L^2(0, 1)$, there exists a control $v \in L^2(0, T; L^2(\omega))$ that ε -desensitizes Φ ; this result is established for all $T \geq 4$, and for arbitrarily chosen nonvoid open sets \mathcal{O} and ω in $(0, 1)$. It is worth

noting that all the results in all the earlier works (except for [12]) dealing with parabolic equations are established under the constraint that the intersection of the observation region \mathcal{O} , and the control region ω is nonempty. The proof technique developed in [5] critically relies on the fact that the one-dimensional wave equation is time periodic. We note that the time periodicity is lost when one considers space dimensions greater than one, or even if staying in the one-dimensional setting, one chooses nonconstant coefficients or even the wave equation with a potential having a constant coefficient; thus, the proof technique developed in [5] does not generalize to any of those cases; a completely new approach needs to be built up. Whence our interest in exploring what happens in the general case. Although we use the ordinary wave equation to present our method, we do know that it applies to a larger class of hyperbolic equations [13]; the choice we have made is just for simplicity sake. Our main results read:

Theorem 1.1. *Assume that \mathcal{O} and ω are two nonempty open sets in Ω with $\mathcal{O} \cap \omega \neq \emptyset$. There exists a positive time T^* depending only on Ω , \mathcal{O} , and ω such that for every $T > T^*$, for all $y^0 \in H_0^1(\Omega)$ and $y^1 \in L^2(\Omega)$, and for every positive constant ε , there exists a control $v \in L^2(0, T; L^2(\omega))$ that ε -desensitizes the functional Φ .*

Theorem 1.2. *Assume that \mathcal{O} and ω are two neighborhoods of Γ_+ , and $x_0 \notin \bar{\Omega}$. There exists a positive time T_0^* depending only on Ω and ω such that for every $T > T_0^*$, and for all $y^0 \in L^2(\Omega)$ and $y^1 \in H^{-1}(\Omega)$, there exists a control $v \in [H^1(0, T; L^2(\omega))]'$ that desensitizes the functional Φ . Moreover, there exists a positive constant C independent of the initial data such that:*

$$\|v\|_{[H^1(0, T; L^2(\omega))]' } \leq C (\|y^0\|_{L^2(\Omega)} + \|y^1\|_{H^{-1}(\Omega)} + \|\xi\|_{L^2(Q)}). \tag{5}$$

Remark 1.3. A precise structure of the constant T^* may be found in [8]. It should be noted that our method, when specialized to the interval $(0, 1)$ with $\omega = (l_1, l_2)$, works for all $T > 2 \max(l_1, 1 - l_2)$; in this case, Theorem 1 improves Theorem 7 in [5] where it is required that $T \geq 4$. A precise structure of the constant T_0^* may be found in [6].

Remark 1.4. It is worth emphasizing that, unlike the one-dimensional wave equation for which the observation set \mathcal{O} and the control set ω can be chosen arbitrarily, for higher dimensional hyperbolic equations, both sets must satisfy the Bardos–Lebeau–Rauch [1] geometric control condition (GCC) for the existence of desensitizing controls to be established. The choice made in Theorem 1.2 represents a special case where (GCC) is met (e.g. [9,11]). Note that \mathcal{O} and ω being neighborhoods of Γ_+ , they automatically satisfy $\mathcal{O} \cap \omega \neq \emptyset$, since $\mathcal{O} \cap \omega$ is also a neighborhood of Γ_+ . However for the existence of ε -desensitizing controls for the wave equation in higher dimensions, \mathcal{O} and ω may be chosen arbitrarily provided that they satisfy $\mathcal{O} \cap \omega \neq \emptyset$, which is the same condition found in all earlier results dealing with the heat equation. It should be noted that the time periodicity property of the one-dimensional wave equation with no sources enabled the author of [5] to get rid of the nonempty intersection constraint; whether this finding is peculiar to the one-dimensional wave equation is yet to be proven (see [12] for a special case involving the 1D heat equation).

2. Basic ideas for proving Theorems 1.1 and 1.2

The main ingredient for proving the existence of desensitizing (resp. ε -desensitizing) controls is to reduce the problem to a (resp. an approximate) controllability problem. To this end, consider the following cascade controlled wave equations:

$$\begin{cases} y_{0tt} - \Delta y_0 = \xi + v\chi_\omega & \text{in } Q, \\ y_0 = 0 & \text{on } \Sigma = \partial\Omega \times (0, T), \\ y_0(0) = y^0; \quad y_{0t}(0) = y^1 & \text{in } \Omega, \end{cases} \tag{6}$$

$$\begin{cases} q_{tt} - \Delta q = y_0\chi_{\mathcal{O}} & \text{in } Q, \\ q = 0 & \text{on } \Sigma = \partial\Omega \times (0, T), \\ q(T) = 0; \quad q_t(T) = 0 & \text{in } \Omega, \end{cases} \tag{7}$$

where y^0 and y^1 are the same as in the theorems above. We have:

Proposition 2.1. (See [5].) (i) A control v ε -desensitizes the functional Φ if and only if the solution pair (y_0, q) of (6)–(7) satisfies:

$$\|q(0)\|_{L^2(\Omega)} \leq \varepsilon, \quad \|q_t(0)\|_{H^{-1}(\Omega)} \leq \varepsilon. \quad (8)$$

(ii) A control v desensitizes the functional Φ if and only if the solution pair (y_0, q) of (6)–(7) satisfies:

$$q(0) = 0, \quad q_t(0) = 0. \quad (9)$$

Remark 2.2. Proposition 2.1 reduces the proof of Theorem 1.1 to showing that system (6)–(7) is approximately controllable, and the proof of Theorem 1.2 to showing that system (6)–(7) is exactly controllable.

Introduce the adjoint system to (6)–(7):

$$\begin{cases} p_{tt} - \Delta p = 0 & \text{in } Q, \\ p = 0 & \text{on } \Sigma = \partial\Omega \times (0, T), \\ p(0) = p^0; \quad p_t(0) = p^1 & \text{in } \Omega, \end{cases} \quad (10)$$

$$\begin{cases} z_{tt} - \Delta z = p\chi_{\mathcal{O}} & \text{in } Q, \\ z = 0 & \text{on } \Sigma = \partial\Omega \times (0, T), \\ z(T) = 0; \quad z_t(T) = 0 & \text{in } \Omega. \end{cases} \quad (11)$$

For $(p^0, p^1) \in H_0^1(\Omega) \times L^2(\Omega)$, we have $p \in C([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$, and $z \in C([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$. For every $t \in [0, T]$, set

$$E(p; t) = \frac{1}{2} \int_{\Omega} \{|p_t(x, t)|^2 + |\nabla p(x, t)|^2\} dx, \quad \widehat{E}(p; t) = \frac{1}{2} (\|p(\cdot, t)\|_{L^2(\Omega)}^2 + \|p_t(\cdot, t)\|_{H^{-1}(\Omega)}^2).$$

Sketch of the proof of Theorem 1.2. The proof of Theorem 1.2 essentially relies on

Proposition 2.3. Let \mathcal{O} , ω , and T be given as in Theorem 1.2. There exists $C_1 > 0$ such that:

$$\widehat{E}(p; 0) \leq C_1 \int_0^T \int_{\omega} |z_t(x, t)|^2 dx dt, \quad \forall (p^0, p^1) \in L^2(\Omega) \times H^{-1}(\Omega). \quad (12)$$

Let us postpone the proof sketch of that proposition for the time being. Introduce the functional

$$\begin{aligned} \mathcal{J} : L^2(\Omega) \times H^{-1}(\Omega) &\longrightarrow \mathbb{R} \\ (p^0, p^1) &\mapsto \mathcal{J}(p^0, p^1) = \frac{1}{2} \int_0^T \int_{\omega} |z_t(x, t)|^2 dx dt + \int_{\Omega} y^0(x) z_t(x, 0) dx \\ &\quad - \langle y^1, z(\cdot, 0) \rangle - \int_{\mathcal{O}} \xi(x, t) z(x, t) dx dt, \end{aligned} \quad (13)$$

where $\langle \cdot, \cdot \rangle$ denotes the duality product between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. It is not difficult to check that \mathcal{J} is strictly convex, and continuous. Further, \mathcal{J} is coercive thanks to Proposition 2.3. Therefore, \mathcal{J} has a unique minimizer (\hat{p}^0, \hat{p}^1) , and if \hat{z} is the associated solution of (11), then we have the Euler equation:

$$\int_0^T \int_{\omega} \hat{z}_t(x, t) z_t(x, t) dx dt + \int_{\Omega} y^0(x) z_t(x, 0) dx - \langle y^1, z(\cdot, 0) \rangle - \int_{\mathcal{O}} \xi(x, t) z(x, t) dx dt = 0, \quad (14)$$

for every z solution of (11). On the other hand, we have the duality identity:

$$\begin{aligned} & \langle p^1, q(\cdot, 0) \rangle - \int_{\Omega} p^0(x)q_t(x, 0) \, dx \\ &= \langle y^1, z(\cdot, 0) \rangle - \int_{\Omega} y^0(x)z_t(x, 0) \, dx + \int_0^T \int_{\omega} v(x, t)z(x, t) \, dx \, dt + \int_Q \xi(x, t)z(x, t) \, dx \, dt. \end{aligned} \tag{15}$$

Choosing the control $v = \hat{z}_{tt} \in [H^1(0, T; L^2(\Omega))]'$ in (6), we easily derive from (14) and (15):

$$\langle p^1, q(\cdot, 0) \rangle - \int_{\Omega} p^0(x)q_t(x, 0) \, dx = 0, \quad \forall (p^0, p^1) \in L^2(\Omega) \times H^{-1}(\Omega), \tag{16}$$

hence (9). This shows that $v = \hat{z}_{tt}$ desensitizes the functional Φ , thanks to Proposition 2.1. It remains to show that v satisfies (5); this is easily done by setting $z = \hat{z}$ in (15), and noticing that: $\|\hat{z}_{tt}\|_{[H^1(0,T;L^2(\Omega))]' } \leq \|\hat{z}_t\|_{L^2(Q)}$. \square

Sketch of the proof of Proposition 2.3. Let ω_0 and ω_1 be two neighborhoods of Γ_+ with $\omega_0 \subset \omega_1 \Subset \mathcal{O} \cap \omega$. Let $r \in C_0^\infty(0, T)$ denote the cut-off function defined in [6, (2.33)], and set $\tilde{p} = rp$. Applying Theorem 2.4 of [6] with $u = \tilde{p}$, we derive the existence of positive constants C, μ , and $\lambda_0 \geq 1$ such that for all $\lambda > \lambda_0$:

$$\int_{T_0}^{T'_0} \int_{\Omega} |p(x, t)|^2 \, dx \, dt \leq C\lambda e^{-\mu\lambda} \widehat{E}(p; 0) + C e^{C\lambda} \int_0^T \int_{\omega_0} r^2 |p(x, t)|^2 \, dx \, dt, \tag{17}$$

where, here and in the sequel, the positive constant C may bear different values, and the constants T_0 and T'_0 are given by [6, (2.29)]. On the other hand one readily checks that:

$$\widehat{E}(p; 0) \leq C \int_{T_0}^{T'_0} \int_{\Omega} |p(x, t)|^2 \, dx \, dt. \tag{18}$$

The combination of (17) and (18) yields for λ large enough:

$$\widehat{E}(p; 0) \leq C \int_0^T \int_{\omega_0} r^2 |p(x, t)|^2 \, dx \, dt. \tag{19}$$

Introduce the function η , which satisfies: $\eta \in C^\infty(\bar{\Omega})$, $0 \leq \eta \leq 1$, $\eta = 1$ in ω_0 , $\eta = 0$, in $\Omega \setminus \omega_1$, $\frac{|\nabla\eta|^2}{\eta} \in C^0(\bar{\Omega})$, $\frac{|\Delta\eta|^2}{\eta} \in C^0(\bar{\Omega})$.

The function $\check{z} = r^2\eta z$ satisfies homogeneous boundary, initial, and final conditions together with the equation:

$$\check{z}_{tt} - \Delta\check{z} = 2\eta(r'^2z + rr''z + 2rr'z_t) - r^2(2\nabla\eta \cdot \nabla z + z\Delta\eta) + r^2\eta p \chi_{\mathcal{O}} \quad \text{in } Q. \tag{20}$$

Multiplying the first equation of (10) by \check{z} , and integrating by parts over Q , we derive (with $\omega_{1T} = \omega_1 \times (0, T)$):

$$\begin{aligned} \int_{\omega_{1T}} r^2\eta |p|^2 \, dx \, dt &= -4 \int_{\omega_{1T}} rr'\eta z_t p \, dx \, dt + \int_{\omega_{1T}} \{-2\eta(rr'' + |r'|^2)zp + r^2(2\nabla\eta \cdot \nabla z + z\Delta\eta)p\} \, dx \, dt \\ &\leq \varepsilon \int_{\omega_{1T}} r^2\eta |p|^2 \, dx \, dt + C_\varepsilon \int_{\omega_{1T}} \{|z|^2 + |z_t|^2 + r^2|\nabla z|^2\} \, dx \, dt, \quad \forall \varepsilon > 0. \end{aligned} \tag{21}$$

Choosing $\varepsilon = 1/2$, we draw from (21):

$$\int_0^T \int_{\omega_0} r^2 |p|^2 \, dx \, dt \leq C \int_{\omega_{1T}} \{|z|^2 + |z_t|^2\} \, dx \, dt + C \int_{\omega_{1T}} r^2 |\nabla z|^2 \, dx \, dt. \tag{22}$$

Now we are going to absorb the utmost right term in (22). To this end, introduce the function ζ , which satisfies $\zeta \in C^\infty(\bar{\Omega})$, $0 \leq \zeta \leq 1$, $\zeta = 1$ in ω_1 , $\zeta = 0$, in $\Omega \setminus (\mathcal{O} \cap \omega)$. The function $\tilde{z} = r^2 \zeta z$ satisfies homogeneous boundary, initial, and final conditions together with Eq. (20) where η is replaced with ζ . Multiplying the latter equation by z and integrating by parts over Q , we find:

$$-\int_Q \tilde{z}_t z_t \, dx \, dt + \int_Q \nabla \tilde{z} \cdot \nabla z \, dx \, dt = \int_Q \{4rr' \zeta z_t z + 2\zeta (rr'' + |r'|^2) z^2 - r^2 (2z \nabla \zeta \cdot \nabla z + z^2 \Delta \zeta)\} \, dx \, dt + \int_Q r^2 \zeta z p \chi_{\mathcal{O}} \, dx \, dt. \quad (23)$$

It follows from (23), integrating by parts where needed, that for every $\delta > 0$:

$$\int_{\omega_{1T}} r^2 |\nabla z|^2 \, dx \, dt \leq C_\delta \int_0^T \int_\omega \{|z|^2 + |z_t|^2\} \, dx \, dt + 2\delta T \widehat{E}(p; 0). \quad (24)$$

Combining (19), (22), (24), choosing $\delta = 1/4CT$ (with C as the product of the constants in (19) and (22)), and using the Poincaré inequality (as $z(T) = 0$), we get the claimed estimate (12). \square

For the proof of Theorem 1.1, we refer the interested reader to Propositions 3 and 4 in [5], and to Theorem 1.1.1 in [8].

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