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Partial Differential Equations

# Magnetic Ginzburg–Landau functional with discontinuous constraint

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**Abstract**

This Note reports on results obtained for minimizers of a Ginzburg–Landau functional with discontinuous constraint. These results concern vortex-pinning and boundary conditions for inhomogeneous superconducting samples. ***To cite this article: A. Kachmar, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***


**Résumé**

**Une fonctionnelle de Ginzburg–Landau magnétique avec une contrainte discontinue.** Cette Note rend compte sur des résultats récents obtenus pour les minimiseurs d'une fonctionnelle de Ginzburg–Landau avec une contrainte discontinue. Ces résultats concernent le phénomène de chevillage (pinning) de vortex et les conditions aux limites pour des échantillons supraconducteurs inhomogènes. ***Pour citer cet article : A. Kachmar, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***


## 1. Ginzburg–Landau functional with discontinuous constraint

Some physical experiments deal with superconducting samples subject to non-constant temperatures, or with samples consisting of superconducting materials with different critical temperatures; see [10] for a recent review concerning these experiments. Such superconducting samples are of particular interest since they permit to increase (or decrease) the value of the onset field $H_{C_3}$ (third critical field), and they serve in controlling the position of vortices, exhibiting thus a phenomenon known as *vortex-pinning*.

In the framework of the Ginzburg–Landau theory, it is proposed to model the energy of an inhomogeneous superconducting sample by means of the following functional (see [2]):

$$\mathcal{G}_{\varepsilon,H}(\psi,A)=\int_{\Omega}\left(\left|(\nabla-\mathrm{i}A)\psi\right|^2+\frac{1}{2\varepsilon^2}\left(p(x)-|\psi|^2\right)^2+|\operatorname{curl}A-H|^2\right)\mathrm{d}x. \tag{1}$$

Here $\Omega\subset\mathbb{R}^2$ is the 2-D cross section of the sample (assumed to occupy a cylinder of infinite height), $H\geqslant 0$ is the intensity of the applied magnetic field, $\frac{1}{\varepsilon}=\kappa>0$ is the Ginzburg–Landau parameter (a temperature independent

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parameter), and $p(x)$ is a real valued function whose values are determined by the local temperature in the sample. The functional (1) is defined for pairs $(\psi, A)$ in the space $\mathcal{H} = H^1(\Omega; \mathbb{C}) \times H^1(\Omega; \mathbb{R}^2)$.

Lassoued and Mironescu [9] analyzed the functional (1) without a magnetic field (i.e. $H = 0$ and $A = 0$) and when the function $p(x)$ is a step function. Aftalion, Sandier and Serfaty [1] analyzed the functional (1) when the function $p(x)$ is smooth and strictly positive. We analyze the functional (1) in the following case:

$$\Omega = B(0,1), \quad p(x) = 1 \quad \text{in } B(0,R), \quad p(x) = a \quad \text{in } B(0,1) \setminus B(0,R), \tag{2}$$

where $B(0,1)$ denotes the unit disc in $\mathbb{R}^2$, $R \in ]0,1[$ and $a \in \mathbb{R}_+ \setminus \{1\}$ are given constants.

## 2. The case without a magnetic field

The next theorem characterizes the set of minimizers of the functional (1) when there is no applied magnetic field, i.e. $H = 0$.

**Theorem 1.** *Assume that $H = 0$. Up to a gauge transformation, the functional* (1) *admits in the space $\mathcal{H}$ a unique minimizer $(u_\varepsilon, 0)$, where $u_\varepsilon : \Omega \to \mathbb{R}$ is a non-negative function.*

*Moreover, there exists a constant $\varepsilon_0$ such that for all $\varepsilon \in ]0, \varepsilon_0[$, $u_\varepsilon \in C^2(\overline{B(0,R)}) \cup C^2(\overline{B(0,1) \setminus B(0,R)})$ and*

$$\min(1, \sqrt{a}) < u_\varepsilon < \max(1, \sqrt{a}) \quad \text{in } \overline{\Omega}.$$

To understand the asymptotic behavior of the function $u_\varepsilon$, we show that there is a unique positive and bounded function $U : \mathbb{R} \to \mathbb{R}$ that solves the equation:

$$-U''(t) = \big(p_0(t) - U^2(t)\big)U(t) \quad \text{in } \mathbb{R}, \quad \text{where } p_0(t) = 1 \text{ in } \mathbb{R}_- \text{ and } p_0(t) = a \text{ in } \mathbb{R}_+. \tag{3}$$

The expression of $U(t)$ can be given explicitly (see [5]), but we only need to know that the quantity $\gamma(a) = U'(0)/U(0)$ is positive when $a < 1$, and negative when $a > 1$.

By a blow-up argument, we are able to describe the asymptotic behavior of the function $u_\varepsilon$ by means of the one-dimensional function $U$.

**Theorem 2.** *The following asymptotic limits hold as $\varepsilon \to 0$:*

$$\lim_{\varepsilon \to 0} \left\| u_\varepsilon(x) - U\left(\frac{|x| - R}{\varepsilon}\right) \right\|_{L^\infty(\Omega)} = 0, \tag{4}$$

$$\forall C > 0, \quad \lim_{\varepsilon \to 0} \varepsilon \left\| u_\varepsilon(x) - U\left(\frac{|x| - R}{\varepsilon}\right) \right\|_{W^{1,\infty}(\{x \in \mathbb{R}^2:\ |R - |x|| \leqslant C\varepsilon\})} = 0. \tag{5}$$

## 3. Vortex pinning

We return to the analysis of minimizers of the functional (1) in the presence of an applied magnetic field, i.e. $H > 0$.

**Theorem 3.** *Let $(\psi_{\varepsilon,H}, A_{\varepsilon,H})$ be a minimizer of* (1). *There exists a constant $a_0 \in ]0,1[$, and for each $a \in ]0, a_0[$, there exist positive constants $\mu_*$, $\mu_\#$, $\varepsilon_0$ and a function $]0, \varepsilon_0[ \ni \varepsilon \mapsto k_\varepsilon \in \mathbb{R}_+$, $0 < \liminf_{\varepsilon \to 0} k_\varepsilon \leqslant \limsup_{\varepsilon \to 0} k_\varepsilon < \infty$, such that*:

(i) *If $H < k_\varepsilon |\ln \varepsilon| - \mu_* \ln|\ln \varepsilon|$, then $|\psi_{\varepsilon,H}| \geqslant \frac{\sqrt{a}}{2}$ in $\overline{\Omega}$.*

(ii) *If $H = k_\varepsilon |\ln \varepsilon| + \mu \ln|\ln \varepsilon|$ and $\mu \geqslant -\mu_*$, then there exists a finite family of balls $(B(a_i(\varepsilon), r_i(\varepsilon)))_i$ with the following properties*:

   (a) $\sum_i r_i(\varepsilon) < |\ln \varepsilon|^{-10}$;

   (b) $|\psi_{\varepsilon,H}| \geqslant \frac{\sqrt{a}}{2}$ *in* $\overline{\Omega} \setminus \bigcup_i B(a_i(\varepsilon), r_i(\varepsilon))$;

   (c) *Letting $d_i$ be the degree of $\psi_{\varepsilon,H}/|\psi_{\varepsilon,H}|$ on $\partial B(a_i(\varepsilon), r_i(\varepsilon))$ if $B(a_i, r_i) \subset \Omega$ and* 0 *otherwise, then we have*

$$\sup_{i, |d_i| > 0} \big|R - |a_i(\varepsilon)|\big| \to 0 \quad \text{as } \varepsilon \to 0.$$

(d) *If $\mu > \mu_\#$ there exist positive constants $c$ and $C$ such that*

$$c\ln|\ln\varepsilon| \leqslant \sum_i |d_i| \leqslant C\ln|\ln\varepsilon|.$$

Theorem 3 shows that the position of vortices is strongly dependent on the parameter $a$. When $a = 1$, it is known that near the first critical magnetic field, the vortices are localized near the center of the disc (cf. [11]). However, Theorem 3 states that vortices are localized near the circle $\partial B(0, R)$, i.e. they are pinned towards the less-superconducting side of the sample. This is the *pinning phenomenon* predicted in the physical literature, see e.g. [3].

Let us explain briefly what stands behind the statement of Theorem 3. When we take $\psi = u_\varepsilon$ in (1) and we minimize the resulting functional over $A \in H^1(\Omega;\mathbb{R}^2)$, we get that the minimizer is $\frac{H}{u_\varepsilon^2}\nabla^\perp h_\varepsilon$, where $h_\varepsilon : \Omega \to ]0,1[$ is the solution of the equation:

$$-\operatorname{div}\left(\frac{1}{u_\varepsilon^2}\nabla h_\varepsilon\right) + h_\varepsilon = 0 \quad \text{in } \Omega, \qquad h_\varepsilon = 1 \quad \text{on } \partial\Omega. \tag{6}$$

The constant $k_\varepsilon$ is defined by $k_\varepsilon = \frac{1}{2}(\max_{x\in\overline{\Omega}} \frac{1-h_\varepsilon(x)}{u_\varepsilon^2(x)})^{-1}$. Thanks to our choice of the domain $\Omega$ and the step function $p(x)$, we show that the function $h_\varepsilon(x)$ is radially symmetric and strictly increasing with respect to $|x|$. This permits to show that

$$0 < \liminf_{\varepsilon\to 0} k_\varepsilon \leqslant \limsup_{\varepsilon\to 0} k_\varepsilon < +\infty.$$

Roughly speaking, the analysis of Sandier and Serfaty (cf. [11]) says that near the first critical magnetic field, the vortices of a minimizer of (1) are localized as $\varepsilon \to 0$ near the set $\Lambda_\varepsilon = \{x \in \overline{\Omega}\colon \frac{1-h_\varepsilon(x)}{u_\varepsilon^2(x)} = \frac{1}{2}k_\varepsilon^{-1}\}$. We localize the set $\Lambda_\varepsilon$ by means of a fine semi classical analysis. We obtain when $a$ is sufficiently small that the set $\Lambda_\varepsilon$ consists of a circle $\partial B(0, R_\varepsilon)$, where $R_\varepsilon \in ]R, 1[$ has the following asymptotic behavior:

$$\varepsilon \ll R_\varepsilon - R \ll \varepsilon^\alpha \quad \text{as } \varepsilon \to 0, \quad (\alpha \in ]0,1[ \text{ is given}).$$

Let us mention that when $a > 1$, we prove that the set $\Lambda_\varepsilon$ consists of a single point, $\Lambda_\varepsilon = \{0\}$, and minimizers of (1) exhibit the same behavior as the one present in [11]. Near the first critical magnetic field, a minimizer has a finite number of vortices localized near the center of the disc and whose exact positions are determined by a finite dimensional problem (a renormalized energy).

## 4. Boundary conditions

**Theorem 4.** *There exists a function $\mathbb{R}_+ \setminus \{1\} \ni a \mapsto \gamma(a) \in \mathbb{R}\setminus\{0\}$ such that, if $(\psi_{\varepsilon,H}, A_{\varepsilon,H})$ is a minimizer of* (1) *satisfying $|\psi_{\varepsilon,H}| > 0$ in $\overline{\Omega}$, then the following limit holds*:

$$\lim_{\varepsilon\to 0} \varepsilon \left\| n(x)\cdot(\nabla - \mathrm{i}A_{\varepsilon,H})\psi_{\varepsilon,H} + \frac{\gamma(a)}{\varepsilon}\psi_{\varepsilon,H}\right\|_{L^2(\partial B(0,R))} = 0, \tag{7}$$

*where $n(x) = \frac{x}{|x|}$ for all $x \in \mathbb{R}^2 \setminus \{0\}$, is the outward unit normal vector.*

*Furthermore, the function $\gamma$ satisfies*: (1) $\gamma(a) > 0$ *if* $a < 1$; (2) $\gamma(a) < 0$ *if* $a > 1$.

When $H = 0$, Theorem 2 provides a stronger result than that of Theorem 4, since the convergence holds in $L^\infty$-norm. Theorem 4 holds, in particular, when the magnetic field satisfies the first regime of Theorem 3. In other words, Theorem 4 states that, in the absence of vortices, minimizers approximately satisfy a Robin type boundary condition as $\varepsilon \to 0$:

$$n(x)\cdot(\nabla - \mathrm{i}A_{\varepsilon,H})\psi_{\varepsilon,H} + \frac{\gamma(a)}{\varepsilon}\psi_{\varepsilon,H} \sim 0 \quad \text{on } \partial B(0,R). \tag{8}$$

This type of boundary condition has been introduced in the context of superconductivity by the physicist de Gennes (see [4]). The parameter $\gamma(a)$ is called the de Gennes parameter. The fact that the parameter $\gamma(a)$ can be negative is in accordance with the physical literature: negative values of the de Gennes parameter were previously proposed to model surface enhancement, see [10].

We would like to conclude by mentioning that the result of Theorem 4 complements results in this direction obtained in [6,7], and justifies the analysis we carried out in [8] for problems involving boundary conditions of the type (8).

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