Abstract

This Note reports on results obtained for minimizers of a Ginzburg–Landau functional with discontinuous constraint. These results concern vortex-pinning and boundary conditions for inhomogeneous superconducting samples. To cite this article: A. Kachmar, C. R. Acad. Sci. Paris, Ser. I 346 (2008).

Résumé


1. Ginzburg–Landau functional with discontinuous constraint

Some physical experiments deal with superconducting samples subject to non-constant temperatures, or with samples consisting of superconducting materials with different critical temperatures; see [10] for a recent review concerning these experiments. Such superconducting samples are of particular interest since they permit to increase (or decrease) the value of the onset field \( H_{C_3} \) (third critical field), and they serve in controlling the position of vortices, exhibiting thus a phenomenon known as vortex-pinning.

In the framework of the Ginzburg–Landau theory, it is proposed to model the energy of an inhomogeneous superconducting sample by means of the following functional (see [2]):

\[
\mathcal{G}_{\varepsilon, H}(\psi, A) = \int_{\Omega} \left( \left| (\nabla - iA)\psi \right|^2 + \frac{1}{2\varepsilon^2} \left( p(x) - |\psi|^2 \right)^2 + |\text{curl} A - H|^2 \right) \, dx.
\]  

(1)

Here \( \Omega \subset \mathbb{R}^2 \) is the 2-D cross section of the sample (assumed to occupy a cylinder of infinite height), \( H \geq 0 \) is the intensity of the applied magnetic field, \( \frac{1}{\varepsilon} = \kappa > 0 \) is the Ginzburg–Landau parameter (a temperature independent
parameter), and \( p(x) \) is a real valued function whose values are determined by the local temperature in the sample. The functional (1) is defined for pairs \( (\psi, A) \) in the space \( \mathcal{H} = H^1(\Omega; \mathbb{C}) \times H^1(\Omega; \mathbb{R}^2) \).

Lassoued and Mironescu [9] analyzed the functional (1) without a magnetic field (i.e. \( H = 0 \) and \( A = 0 \)) and when the function \( p(x) \) is a step function. Aftalion, Sandier and Serfaty [1] analyzed the functional (1) when the function \( p(x) \) is smooth and strictly positive. We analyze the functional (1) in the following case:

\[
\Omega = B(0, 1), \quad p(x) = 1 \quad \text{in} \quad B(0, R), \quad p(x) = a \quad \text{in} \quad B(0, 1) \setminus B(0, R),
\]

(2)

where \( B(0, 1) \) denotes the unit disc in \( \mathbb{R}^2, R \in ]0, 1[ \) and \( a \in \mathbb{R}_+ \setminus \{1\} \) are given constants.

2. The case without a magnetic field

The next theorem characterizes the set of minimizers of the functional (1) when there is no applied magnetic field, i.e. \( H = 0 \).

**Theorem 1.** Assume that \( H = 0 \). Up to a gauge transformation, the functional (1) admits in the space \( \mathcal{H} \) a unique minimizer \( (u_\varepsilon, 0) \), where \( u_\varepsilon : \Omega \rightarrow \mathbb{R} \) is a non-negative function.

Moreover, there exists a constant \( \varepsilon_0 \) such that for all \( \varepsilon \in ]0, \varepsilon_0[ \), \( u_\varepsilon \in C^2(B(0, R)) \cup C^2(B(0, 1) \setminus B(0, R)) \) and

\[
\min(1, \sqrt{a}) < u_\varepsilon < \max(1, \sqrt{a}) \quad \text{in} \quad \overline{\Omega}.
\]

To understand the asymptotic behavior of the function \( u_\varepsilon \), we show that there is a unique positive and bounded function \( U : \mathbb{R} \rightarrow \mathbb{R} \) that solves the equation:

\[
-U''(t) = \left( p_0(t) - U^2(t) \right) U(t) \quad \text{in} \quad \mathbb{R}, \quad \text{where} \quad p_0(t) = 1 \quad \text{in} \quad \mathbb{R}_- \quad \text{and} \quad p_0(t) = a \quad \text{in} \quad \mathbb{R}_+.
\]

(3)

The expression of \( U(t) \) can be given explicitly (see [5]), but we only need to know that the quantity \( \gamma(a) = U'(0)/U(0) \) is positive when \( a < 1 \), and negative when \( a > 1 \).

By a blow-up argument, we are able to describe the asymptotic behavior of the function \( u_\varepsilon \) by means of the one-dimensional function \( U \).

**Theorem 2.** The following asymptotic limits hold as \( \varepsilon \rightarrow 0 \):

\[
\lim_{\varepsilon \rightarrow 0} || u_\varepsilon(x) - U\left(\left|\frac{x}{\varepsilon}\right| - \frac{R}{\varepsilon}\right) ||_{L^\infty(\Omega)} = 0,
\]

(4)

\[
\forall C > 0, \quad \lim_{\varepsilon \rightarrow 0} \left\| u_\varepsilon(x) - U\left(\left|\frac{x}{\varepsilon}\right| - \frac{R}{\varepsilon}\right) \right\|_{W^{1, \infty(\{|x| \leq C\varepsilon\})}} = 0.
\]

(5)

3. Vortex pinning

We return to the analysis of minimizers of the functional (1) in the presence of an applied magnetic field, i.e. \( H > 0 \).

**Theorem 3.** Let \( (\psi_{\varepsilon, H}, A_{\varepsilon, H}) \) be a minimizer of (1). There exists a constant \( a_0 \in ]0, 1[ \), and for each \( a \in ]0, a_0[ \), there exist positive constants \( \mu_+, \mu_-, \varepsilon_0 \) and a function \( ]0, \varepsilon_0[ \ni \varepsilon \mapsto k_\varepsilon \in \mathbb{R}_+ \), \( 0 < \liminf_{\varepsilon \rightarrow 0} k_\varepsilon \leq \limsup_{\varepsilon \rightarrow 0} k_\varepsilon < \infty \), such that:

(i) If \( H < k_\varepsilon \left| \ln \varepsilon \right| - \mu_+ \ln \left| \ln \varepsilon \right| \), then \( |\psi_{\varepsilon, H}| > \frac{\sqrt{a}}{2} \) in \( \overline{\Omega} \).

(ii) If \( H = k_\varepsilon \left| \ln \varepsilon \right| + \mu \ln \left| \ln \varepsilon \right| \) and \( \mu > -\mu_+ \), then there exists a finite family of balls \( (B(a_i(\varepsilon), r_i(\varepsilon))) \), with the following properties:

(a) \( \sum_i r_i(\varepsilon) < \left| \ln \varepsilon \right|^{-10} \),

(b) \( |\psi_{\varepsilon, H}| > \frac{\sqrt{a}}{2} \) in \( \overline{\Omega} \setminus \bigcup B(a_i(\varepsilon), r_i(\varepsilon)) \);

(c) Letting \( d_i \) be the degree of \( \psi_{\varepsilon, H}/|\psi_{\varepsilon, H}| \) on \( \partial B(a_i(\varepsilon), r_i(\varepsilon)) \) if \( B(a_i, r_i) \subset \Omega \) and 0 otherwise, then we have

\[
\sup_{i, |d_i| > 0} \left( R - |a_i(\varepsilon)| \right) \rightarrow 0 \quad \text{as} \quad \varepsilon \rightarrow 0.
\]
(d) If $\mu > \mu_\#$, there exist positive constants $c$ and $C$ such that
\[ c \ln |\ln \epsilon| \lesssim \sum_i |d_i| \lesssim C \ln |\ln \epsilon|. \]

Theorem 3 shows that the position of vortices is strongly dependent on the parameter $a$. When $a = 1$, it is known that near the first critical magnetic field, the vortices are localized near the center of the disc (cf. [11]). However, Theorem 3 states that vortices are localized near the circle $\partial B(0, R)$, i.e. they are pinned towards the less-superconducting side of the sample. This is the pinning phenomenon predicted in the physical literature, see e.g. [3].

Let us explain briefly what stands behind the statement of Theorem 3. When we take $\psi = u_\epsilon$ in (1) and we minimize the resulting functional over $A \in H^1(\Omega; \mathbb{R}^2)$, we get that the minimizer is $\frac{H}{u_\epsilon^2} \nabla h_\epsilon$, where $h_\epsilon : \Omega \to \mathbb{R}$ is the solution of the equation:
\[-\text{div} \left( \frac{1}{u_\epsilon^2} \nabla h_\epsilon \right) + h_\epsilon = 0 \text{ in } \Omega, \quad h_\epsilon = 1 \text{ on } \partial \Omega. \tag{6} \]

The constant $k_\epsilon$ is defined by $k_\epsilon = \frac{1}{2} \left( \max_{x \in \Omega} \left( \frac{1}{u_\epsilon^2(x)} - \frac{1}{a^2(x)} \right) \right)^{-1}$. Thanks to our choice of the domain $\Omega$ and the step function $p(x)$, we show that the function $h_\epsilon(x)$ is radially symmetric and strictly increasing with respect to $|x|$. This permits to show that
\[ 0 < \liminf_{\epsilon \to 0} k_\epsilon \leq \limsup_{\epsilon \to 0} k_\epsilon < +\infty. \]

Roughly speaking, the analysis of Sandier and Serfaty (cf. [11]) says that near the first critical magnetic field, the vortices of a minimizer of (1) are localized as $\epsilon \to 0$ near the set $A_\epsilon = \{ x \in \Omega : |h_\epsilon(x)| = \frac{1}{2} k_\epsilon^{-1} \}$. We localize the set $A_\epsilon$ by means of a fine semi classical analysis. We obtain when $a$ is sufficiently small that the set $A_\epsilon$ consists of a circle $\partial B(0, R_\epsilon)$, where $R_\epsilon \in ]R, 1[$ has the following asymptotic behavior:
\[ \epsilon \ll R_\epsilon - R \ll \epsilon^a \quad \text{as } \epsilon \to 0, \quad (a \in ]0, 1[ \text{ is given}) . \]

Let us mention that when $a > 1$, we prove that the set $A_\epsilon$ consists of a single point, $A_\epsilon = \{ 0 \}$, and minimizers of (1) exhibit the same behavior as the one present in [11]. Near the first critical magnetic field, a minimizer has a finite number of vortices localized near the center of the disc and whose exact positions are determined by a finite dimensional problem (a renormalized energy).

4. Boundary conditions

**Theorem 4.** There exists a function $\mathbb{R}_+ \setminus \{ 1 \} \ni a \mapsto \gamma(a) \in \mathbb{R} \setminus \{ 0 \}$ such that, if $(\psi_{\epsilon, H}, A_{\epsilon, H})$ is a minimizer of (1) satisfying $|\psi_{\epsilon, H}| > 0$ in $\Omega$, then the following limit holds:
\[ \lim_{\epsilon \to 0} \epsilon \left\| n(x) \cdot (\nabla - iA_{\epsilon, H}) \psi_{\epsilon, H} + \frac{\gamma(a)}{\epsilon} \psi_{\epsilon, H} \right\|_{L^2(\partial B(0, R))} = 0, \tag{7} \]
where $n(x) = \frac{x}{|x|}$ for all $x \in \mathbb{R}^2 \setminus \{ 0 \}$, is the outward unit normal vector.

Furthermore, the function $\gamma$ satisfies: (1) $\gamma(a) > 0$ if $a < 1$; (2) $\gamma(a) < 0$ if $a > 1$.

When $H = 0$, Theorem 2 provides a stronger result than that of Theorem 4, since the convergence holds in $L^\infty$-norm. Theorem 4 holds, in particular, when the magnetic field satisfies the first regime of Theorem 3. In other words, Theorem 4 states that, in the absence of vortices, minimizers approximately satisfy a Robin type boundary condition as $\epsilon \to 0$:
\[ n(x) \cdot (\nabla - iA_{\epsilon, H}) \psi_{\epsilon, H} + \frac{\gamma(a)}{\epsilon} \psi_{\epsilon, H} \sim 0 \quad \text{on } \partial B(0, R). \tag{8} \]

This type of boundary condition has been introduced in the context of superconductivity by the physicist de Gennes (see [4]). The parameter $\gamma(a)$ is called the de Gennes parameter. The fact that the parameter $\gamma(a)$ can be negative is in accordance with the physical literature: negative values of the de Gennes parameter were previously proposed to model surface enhancement, see [10].
We would like to conclude by mentioning that the result of Theorem 4 complements results in this direction obtained in [6,7], and justifies the analysis we carried out in [8] for problems involving boundary conditions of the type (8).

References