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The Boltzmann–Grad limit of the periodic Lorentz gas in two space dimensions

Emanuele Caglioti ^a, François Golse ^b

^a Università di Roma “La Sapienza”, Dipartimento di Matematica “Guido Castelnuovo”, P.le Aldo Moro 2, 00185 Rome, Italy

^b École polytechnique, centre de mathématiques Laurent-Schwartz, 91128 Palaiseau cedex, France

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Abstract

The periodic Lorentz gas is the dynamical system corresponding to the free motion of a point particle in a periodic system of fixed spherical obstacles of radius r centered at the integer points of the Euclidian plane, assuming all collisions of the particle with the obstacles to be elastic. In this Note, we study this motion on time intervals of order $1/r$ as $r \rightarrow 0^+$. **To cite this article:** E. Caglioti, F. Golse, C. R. Acad. Sci. Paris, Ser. I 346 (2008).

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Résumé

La limite de Boltzmann–Grad du gaz de Lorentz périodique en dimension deux d'espace. Le gaz de Lorentz périodique est le système dynamique correspondant au mouvement libre dans le plan d'une particule ponctuelle rebondissant de manière élastique sur un système de disques de rayon r centrés aux points de coordonnées entières. On étudie ce mouvement pour $r \rightarrow 0^+$ sur des temps de l'ordre de $1/r$. **Pour citer cet article :** E. Caglioti, F. Golse, C. R. Acad. Sci. Paris, Ser. I 346 (2008).

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On appelle gaz de Lorentz le système dynamique correspondant au mouvement libre d'une particule ponctuelle dans un système d'obstacles circulaires de rayon r centrés aux sommets d'un réseau de \mathbf{R}^2 , supposant que les collisions entre la particule et les obstacles sont parfaitement élastiques. Les trajectoires de la particule sont alors données par les formules (2). La limite de Boltzmann–Grad pour le gaz de Lorentz consiste à supposer que le rayon des obstacles $r \rightarrow 0^+$, et à observer la dynamique de la particule sur des plages de temps longues, de l'ordre de $1/r$ — voir (3) pour la loi d'échelle de Boltzmann–Grad en dimension 2.

Or les trajectoires de la particule s'expriment en fonction de l'application de transfert d'obstacle à obstacle T_r définie par (8) — où la notation Y désigne la transformation inverse de (7) — application qui associe, à tout paramètre d'impact $h' \in [-1, 1]$ correspondant à une particule quittant la surface d'un obstacle dans la direction $\omega \in \mathbf{S}^1$, le

E-mail addresses: caglioti@mat.uniroma1.it (E. Caglioti), golse@math.polytechnique.fr (F. Golse).

paramètre d'impact h à la collision suivante, ainsi que le temps s s'écoulant jusqu'à cette collision. (Pour une définition de la notion de paramètre d'impact, voir (6).)

On se ramène donc à étudier le comportement limite de l'application de transfert T_r pour $r \rightarrow 0^+$.

Proposition 0.1. *Lorsque $0 < \omega_2 < \omega_1$ et $\alpha = \frac{\omega_2}{\omega_1} \notin \mathbf{Q}$, l'application de transfert T_r est approchée à $O(r^2)$ près par l'application $\mathbf{T}_{A,B,Q,N}$ définie à la formule (14). Pour $\omega \in \mathbf{S}^1$ quelconque tel que $\omega_1\omega_2 \neq 0$ et $\frac{\omega_2}{\omega_1} \notin \mathbf{Q}$, on se ramène au cas ci-dessus par la symétrie (15).*

Les paramètres A, B, Q, N mod. 2 intervenant dans l'application de transfert asymptotique sont définis à partir du développement en fraction continue (9) de α par les formules (11) et (12).

On voit sur ces formules que les paramètres A, B, Q, N mod. 2 sont des fonctions très fortement oscillantes des variables ω et r . Il est donc naturel de chercher le comportement limite de l'application de transfert T_r dans une topologie faible vis à vis de la dépendance en la direction ω . On montre ainsi que, pour tout $h' \in [-1, 1]$, la famille d'applications $\omega \mapsto T_r(h', \omega)$ converge en distribution et au sens de Cesàro lorsque $r \rightarrow 0^+$ vers une mesure de probabilité $P(s, h|h') ds dh$ indépendante de ω :

Théorème 0.2. *Pour tout $\Phi \in C_c(\mathbf{R}_+^* \times]-1, 1[)$ et tout $h' \in]-1, 1[$, la limite (16) a lieu p. p. en $\omega \in \mathbf{S}^1$ lorsque $\varepsilon \rightarrow 0^+$, où la mesure de probabilité $P(s, h|h') ds dh$ est l'image de la probabilité μ définie dans (19) par l'application $(A, B, Q, N) \mapsto \mathbf{T}_{A,B,Q,N}(h')$ de la formule (14). De plus, cette densité de probabilité de transition $P(s, h|h')$ vérifie les propriétés (18).*

Le théorème ci-dessus est le résultat principal de cette Note : il montre que, dans la limite de Boltzmann–Grad, le transfert d'obstacle à obstacle est décrit de manière naturelle par une densité de probabilité de transition $P(s, h|h')$, où s est le laps de temps entre deux collisions successives avec les obstacles (dans l'échelle de temps de la limite de Boltzmann–Grad), h étant le paramètre d'impact lors de la collision future et h' celui correspondant à la collision passée.

Le fait que la probabilité de transition $P(s, h|h')$ soit indépendante de la direction suggère l'hypothèse d'indépendance (H) asymptotique des quantités A, B, Q, N mod. 2 correspondant à des collisions successives.

Théorème 0.3. *Sous l'hypothèse (H), pour toute densité de probabilité $f^{\text{in}} \in C_c(\mathbf{R}^2 \times \mathbf{S}^1)$, la fonction de distribution $f_r \equiv f_r(t, x, \omega)$ de la théorie cinétique, définie par (3) converge dans $L^\infty(\mathbf{R}_+ \times \mathbf{R}^2 \times \mathbf{S}^1)$ faible-* vers la limite (23) lorsque $r \rightarrow 0^+$, où F est la solution du problème de Cauchy (22) posé dans l'espace des phases étendu $(x, \omega, s, h) \in \mathbf{R}^2 \times \mathbf{S}^1 \times \mathbf{R}_+^* \times]-1, 1[$.*

Dans le cas d'obstacles aléatoires indépendants et poissonniens, Gallavotti a montré que la limite de Boltzmann–Grad du gaz de Lorentz obéit à l'équation cinétique de Lorentz (4). Le cas périodique est absolument différent : en se basant sur des estimations (cf. [3] et [8]) du temps de sortie du domaine Z_r défini dans (1), on démontre que la limite de Boltzmann–Grad du gaz de Lorentz périodique ne peut pas être décrite par l'équation de Lorentz (4) sur l'espace des phases $\mathbf{R}^2 \times \mathbf{S}^1$ classique de la théorie cinétique : voir [7]. Si l'hypothèse (H) ci-dessous était vérifiée, le modèle cinétique (23) dans l'espace des phases étendu fournirait donc l'équation devant remplacer l'équation cinétique classique de Lorentz (4) dans le cas périodique.

1. The periodic Lorentz gas

The periodic Lorentz gas is the dynamical system corresponding to the free motion of a single point particle in a periodic system of fixed spherical obstacles, assuming that collisions between the particle and any of the obstacles are elastic. Henceforth, we assume that the space dimension is 2 and that the obstacles are disks of radius r centered at each point of \mathbf{Z}^2 . Hence the domain left free for particle motion is

$$Z_r = \{x \in \mathbf{R}^2 \mid \text{dist}(x, \mathbf{Z}^2) > r\}, \quad \text{where it is assumed that } 0 < r < \frac{1}{2}. \quad (1)$$

Assuming that the particle moves at speed 1, its trajectory starting from $x \in Z_r$ with velocity $\omega \in \mathbf{S}^1$ at time $t = 0$ is $t \mapsto (X_r, \Omega_r)(t; x, \omega) \in \mathbf{R}^2 \times \mathbf{S}^1$ given by

$$\begin{aligned} \dot{X}_r(t) &= \Omega_r(t) & \text{and} \quad \dot{\Omega}_r(t) &= 0 & \text{whenever} \quad X_r(t) \in Z_r, \\ X_r(t+0) &= X_r(t-0) & \text{and} \quad \Omega_r(t+0) &= \mathcal{R}[X_r(t)]\Omega_r(t-0) & \text{whenever} \quad X_r(t-0) \in \partial Z_r, \end{aligned} \quad (2)$$

denoting $\dot{\cdot} = \frac{d}{dt}$ and $\mathcal{R}[X_r(t)]$ the specular reflection on ∂Z_r at the point $X_r(t) = X_r(t \pm 0)$. Assume that the initial position x and direction ω of the particle are distributed in $Z_r \times \mathbf{S}^1$ with some probability density $f^{\text{in}} \equiv f^{\text{in}}(x, \omega)$, and define

$$f_r(t, x, \omega) := f^{\text{in}}(r X_r(-t/r; x, \omega), \Omega_r(-t/r; x, \omega)) \quad \text{whenever } x \in Z_r. \quad (3)$$

We are concerned with the limit of f_r as $r \rightarrow 0^+$ in some appropriate sense to be explained below. In the 2-dimensional setting considered here, this is precisely the Boltzmann–Grad limit.

In the case of a random (Poisson), instead of periodic, configuration of obstacles, Gallavotti [5] proved that the expectation of f_r converges to the solution of the Lorentz kinetic equation for $(x, \omega) \in \mathbf{R}^2 \times \mathbf{S}^1$:

$$(\partial_t + \omega \cdot \nabla_x) f(t, x, \omega) = \int_{\mathbf{S}^1} (f(t, x, \omega - 2(\omega \cdot n)n) - f(t, x, \omega)) (\omega \cdot n)_+ dn, \quad f|_{t=0} = f^{\text{in}}. \quad (4)$$

In the case of a periodic distribution of obstacles, the Boltzmann–Grad limit of the Lorentz gas cannot be described by a transport equation such as (4) see [7] for a complete proof, based on estimates on the free path length to be found in [3] and [8]. This limit involves instead a linear Boltzmann equation on an extended phase space with two new variables taking into account correlations between consecutive collisions with the obstacles that are an effect of periodicity: see Theorem 4.1.

2. The transfer map

Denote by n_x the inward unit normal to Z_r at the point $x \in \partial Z_r$, and consider

$$\Gamma_r^\pm = \{(x, \omega) \in \partial Z_r \times \mathbf{S}^1 \mid \pm \omega \cdot n_x > 0\}, \quad \Gamma_r^0 = \{(x, \omega) \in \partial Z_r \times \mathbf{S}^1 \mid \pm \omega \cdot n_x = 0\}, \quad (5)$$

and let $(\Gamma_r^+ \cup \Gamma_r^0)/\mathbf{Z}^2$ be the quotient of $\Gamma_r^+ \cup \Gamma_r^0$ under the action of \mathbf{Z}^2 by translation on the x variable. For $(x, \omega) \in \Gamma_r^+ \cup \Gamma_r^0$, let $\tau_r(x, \omega)$ be the exit time and $h_r(x, \omega)$ be the impact parameter defined by:

$$\tau_r(x, \omega) = \inf\{t > 0 \mid x + t\omega \in \partial Z_r\}, \quad \text{and} \quad h_r(x, \omega) = \sin(\omega, n_x). \quad (6)$$

Obviously, the map

$$(\Gamma_r^+ \cup \Gamma_r^0)/\mathbf{Z}^2 \ni (x, \omega) \mapsto (h_r(x, \omega), \omega) \in [-1, 1] \times \mathbf{S}^1 \quad (7)$$

coordinatizes $(\Gamma_r^+ \cup \Gamma_r^0)/\mathbf{Z}^2$, and we henceforth denote by Y_r its inverse.

For each $r \in]0, \frac{1}{2}[$, consider now the transfer map $T_r : [-1, 1] \times \mathbf{S}^1 \rightarrow \mathbf{R}_+^* \times [-1, 1]$ defined by

$$T_r(h', \omega) = (r \tau_r(Y_r(h', \omega)), h_r(X_r(\tau_r(Y_r(h', \omega)) \pm 0; Y_r(h', \omega)), \Omega_r(\tau_r(Y_r(h', \omega)) \pm 0; Y_r(h', \omega))). \quad (8)$$

For a particle leaving the surface of an obstacle in the direction ω with impact parameter h' , the transition map $T_r(h', \omega) = (s, h)$ gives the (rescaled) distance s to the next collision, and the corresponding impact parameter h . Obviously, each trajectory (2) of the particle can be expressed in terms of the transfer map T_r and iterates thereof. The Boltzmann–Grad limit of the periodic Lorentz gas is therefore reduced to computing the limiting behavior of T_r as $r \rightarrow 0^+$, and this is our main purpose in this Note.

We first need some pieces of notation. Assume $\omega = (\omega_1, \omega_2)$ with $0 < \omega_2 < \omega_1$, and $\alpha = \omega_2/\omega_1 \in]0, 1[\setminus \mathbf{Q}$. Consider the continued fraction expansion of α :

$$\alpha = [0; a_0, a_1, a_2, \dots] = \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots}}}. \quad (9)$$

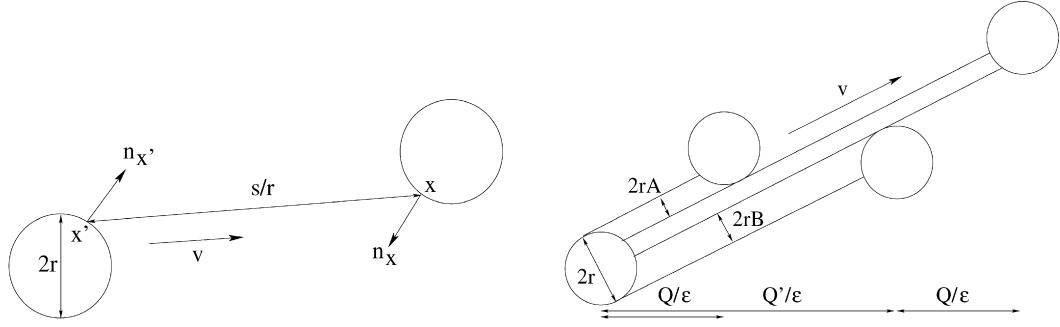


Fig. 1. Left: the transfer map $(s, h) = T_r(h', v)$, with $h' = \sin(n_{x'}, v)$ and $h = \sin(n_x, v)$. Right: Particles leaving the surface of one obstacle will next collide with one of generically three obstacles. The figure explains the geometrical meaning of A, B, Q .

Define the sequences of convergents $(p_n, q_n)_{n \geq 0}$ and errors $(d_n)_{n \geq 0}$ by the recursion formulas

$$\begin{aligned} p_{n+1} &= a_n p_n + p_{n-1}, \quad p_0 = 1, \quad p_1 = 0, \quad d_n = (-1)^{n-1}(q_n \alpha - p_n), \\ q_{n+1} &= a_n q_n + q_{n-1}, \quad q_0 = 0, \quad q_1 = 1, \end{aligned} \quad (10)$$

and let

$$N(\alpha, r) = \inf\{n \geq 0 \mid d_n \leq 2r\sqrt{1+\alpha^2}\}, \quad \text{and} \quad k(\alpha, r) = -\left[\frac{2r\sqrt{1+\alpha^2} - d_{N(\alpha,r)-1}}{d_{N(\alpha,r)}}\right]. \quad (11)$$

Proposition 2.1. For each $\omega = (\cos \theta, \sin \theta)$ with $0 < \theta < \frac{\pi}{4}$, set $\alpha = \tan \theta$ and $\epsilon = 2r\sqrt{1+\alpha^2}$, and

$$A(\alpha, r) = 1 - \frac{d_{N(\alpha,r)}}{\epsilon}, \quad B(\alpha, r) = 1 - \frac{d_{N(\alpha,r)-1} - k(\alpha, r)d_{N(\alpha,r)}}{\epsilon}, \quad Q(\alpha, r) = \epsilon q_{N(\alpha,r)}, \quad (12)$$

(see Fig. 1). In the limit $r \rightarrow 0^+$, the transition map T_r defined in (8) is explicit in terms of A, B, Q, N up to $O(r^2)$:

$$T_r(h', \omega) = \mathbf{T}_{A(\alpha,r), B(\alpha,r), Q(\alpha,r), N(\alpha,r)}(h') + (O(r^2), 0) \quad \text{for each } h' \in [-1, 1]. \quad (13)$$

In the formula above

$$\begin{aligned} \mathbf{T}_{A,B,Q,N}(h') &= (Q, h' - 2(-1)^N(1-A)) && \text{if } (-1)^N h' \in [1-2A, 1], \\ \mathbf{T}_{A,B,Q,N}(h') &= (Q', h' + 2(-1)^N(1-B)) && \text{if } (-1)^N h' \in [-1, -1+2B], \\ \mathbf{T}_{A,B,Q,N}(h') &= (Q' + Q, h' + 2(-1)^N(A-B)) && \text{if } (-1)^N h' \in]-1+2B, 1-2A[, \end{aligned} \quad (14)$$

for each $(A, B, Q, N) \in K := [0, 1]^3 \times \mathbf{Z}/2\mathbf{Z}$, with the notation $Q' = \frac{1-Q(1-B)}{1-A}$.

The proof uses the 3-term partition of the 2-torus defined in Section 2 of [4], following the work of [1].

For $\omega = (\cos \theta, \sin \theta)$ with arbitrary $\theta \in \mathbf{R}$ such that $\cos \theta \neq 0$ and $\tan \theta \notin \mathbf{Q}$, the map $h' \mapsto T_r(h', \omega)$ is computed using Proposition 2.1 in the following manner. Set $\tilde{\theta} = \theta - m\frac{\pi}{2}$ with $m = [\frac{2}{\pi}(\theta + \frac{\pi}{4})]$ and let $\tilde{\omega} = (\cos \tilde{\theta}, \sin \tilde{\theta})$. Then

$$T_r(h', \omega) = (s, h), \quad \text{where } (s, \operatorname{sign}(\tan \tilde{\theta})h) = T_r(\operatorname{sign}(\tan \tilde{\theta})h', \tilde{\omega}). \quad (15)$$

3. The Boltzmann–Grad limit of the transfer map T_r

The formulas (11) and (12) defining A, B, Q, N mod. 2 show that these quantities are strongly oscillating functions of the variables ω and r . In view of Proposition 2.1, one therefore expects the transfer map T_r to have a limit as $r \rightarrow 0^+$ only in the weakest imaginable sense, i.e. in distribution and in the sense of Cesàro.

Theorem 3.1. For each $\Phi \in C_c(\mathbf{R}_+^* \times [-1, 1])$ and each $h' \in [-1, 1]$

$$\frac{1}{|\ln \epsilon|} \int_{-\epsilon}^{1/\epsilon} \Phi(T_r(h', \omega)) \frac{dr}{r} \rightarrow \int_0^\infty \int_{-1}^1 \Phi(s, h) P(s, h|h') ds dh \quad \text{for a.e. } \omega \in \mathbf{S}^1 \text{ as } \epsilon \rightarrow 0^+, \quad (16)$$

where the transition probability $P(s, h|h') \, ds \, dh$ is given by the formula

$$\begin{aligned} P(s, h|h') = \frac{3}{\pi^2 Z} & \left(\left(\left(s - \frac{1}{2} Z \right) \wedge \left(1 + \frac{1}{2} Z \right) - W \vee 1 \right)_+ \right. \\ & \left. + \left(\left(s - \frac{1}{2} Z \right) \wedge 1 - W \vee \left(1 - \frac{1}{2} Z \right) \right)_+ + Z \wedge |1-s| \mathbf{1}_{s<1} + (Z - |1-s|)_+ \right), \end{aligned} \quad (17)$$

with the notations $Z = \frac{1}{2}s|h' - h|$, $W = \frac{1}{2}s + \frac{1}{4}s|h + h'|$, $a \wedge b = \inf(a, b)$ and $a \vee b = \sup(a, b)$. Moreover

$$(s, h, h') \mapsto (1+s)P(s, h|h') \quad \text{is bounded on } \mathbf{R}_+ \times [-1, 1] \times [-1, 1]. \quad (18)$$

The proof of (16)–(19) is based on the explicit representation of the transfer map in Proposition 2.1 and Birkhoff's ergodic theorem for continued fractions, together with Kloosterman sums techniques as in [2]. The probability measure $P(s, h|h') \, ds \, dh$ is found as the image of the probability measure on K given by

$$d\mu(A, B, Q, N) = \frac{6}{\pi^2} \mathbf{1}_{0 < A < 1} \mathbf{1}_{0 < B < 1-A} \mathbf{1}_{0 < Q < \frac{1}{2-A-B}} \frac{dA dB dQ}{1-A} (\delta_{N=0} + \delta_{N=1}) \quad (19)$$

under the map $K \ni (A, B, Q, N) \mapsto \mathbf{T}_{A, B, Q, N}(h') \in \mathbf{R}_+ \times [-1, 1]$.

4. The Boltzmann–Grad limit of the periodic Lorentz gas dynamics

For each $r \in]0, \frac{1}{2}[$, let $d\gamma_r^+(x, \omega)$ be the probability measure on $\Gamma_r^+ \cup \Gamma_r^0$ proportional to $\omega \cdot n_x \, dx \, d\omega$. This probability measure is invariant under the billiard map

$$\mathbf{B}_r : \Gamma_r^+ \cup \Gamma_r^0 \ni (x, \omega) \mapsto \mathbf{B}_r(x, \omega) = (x + \tau_r(x, \omega)\omega, \mathcal{R}[x + \tau_r(x, \omega)\omega]\omega) \in \Gamma_r^+ \cup \Gamma_r^0. \quad (20)$$

For $(x, \omega) \in Z_r \times \mathbf{S}^1$, let $(x^0, \omega^0) = (x - \tau_r(x, -\omega)\omega, \omega)$, and for each $n \geq 0$, set $(x^n, \omega^n) = \mathbf{B}_r^n(x^0, \omega^0)$ and $\alpha^n = \min(|\omega_1^n/\omega_2^n|, |\omega_2^n/\omega_1^n|)$, and define

$$b_r^n = (A(\alpha_n, r), B(\alpha_n, r), Q(\alpha_n, r), N(\alpha_n, r) \bmod. 2) \in K \quad \text{for each } n \geq 0. \quad (21)$$

Assume the existence of a probability measure Π on $\mathbf{R}_+ \times [-1, 1]$ such that, for each $n \geq 1$ and each $\Psi \in C(\mathbf{R}^2 \times \mathbf{S}^1 \times \mathbf{R}_+ \times [-1, 1] \times K^n)$ with compact support

$$\begin{aligned} & \lim_{r \rightarrow 0^+} \int_{Z_r \times \mathbf{S}^1} \Psi\left(x, \omega, r\tau_r\left(\frac{x}{r}, \omega\right), h_r\left(\frac{x_1}{r}, \omega_1\right), b_r^1, \dots, b_r^n\right) dx \, d\omega \\ &= \int_{\mathbf{R}^2 \times \mathbf{S}^1} dx \, d\omega \int_{\mathbf{R}_+ \times [-1, 1]} d\Pi(\tau, h) \int_{K^n} \Psi(x, \omega, \tau, h, \beta_1, \dots, \beta_n) d\mu(\beta_1) \cdots d\mu(\beta_n). \end{aligned} \quad (\text{H})$$

Under this assumption, the Boltzmann–Grad limit of the Lorentz gas is described by a kinetic model on the extended phase space $\mathbf{R}^2 \times \mathbf{S}^1 \times \mathbf{R}_+ \times [-1, 1]$ — unlike the Lorentz kinetic equation (4), that is set on the usual phase space $\mathbf{R}^2 \times \mathbf{S}^1$. The idea of using this extended phase space was proposed for the first time by the authors in [6].

Theorem 4.1. Assume (H), and let f^{in} be any continuous, compactly supported probability density on $\mathbf{R}^2 \times \mathbf{S}^1$. Denoting by $\tilde{R}[\theta]$ the rotation of an angle θ , let $F \equiv F(t, x, \omega, s, h)$ be the solution of

$$\begin{aligned} (\partial_t + \omega \cdot \nabla_x - \partial_s) F(t, x, \omega, s, h) &= \int_{-1}^1 P(s, h|h') F(t, x, \tilde{R}[\pi - 2 \arcsin(h')]\omega, 0, h') dh' \\ F(0, x, \omega, s, h) &= f^{\text{in}}(x, \omega) \int_s^\infty \int_{-1}^1 P(\tau, h|h') dh' d\tau \end{aligned} \quad (22)$$

where (x, ω, s, h) runs through $\mathbf{R}^2 \times \mathbf{S}^1 \times \mathbf{R}_+^* \times [-1, 1]$. Then the family $(f_r)_{0 < r < \frac{1}{2}}$ defined in (3) satisfies

$$f_r \rightarrow \int_0^\infty \int_{-1}^1 F(\cdot, \cdot, \cdot, s, h) ds dh \quad \text{in } L^\infty(\mathbf{R}_+ \times \mathbf{R}^2 \times \mathbf{S}^1) \text{ weak-* as } r \rightarrow 0^+. \quad (23)$$

For each $(s_0, h_0) \in \mathbf{R}_+ \times [-1, 1]$, let $(s_n, h_n)_{n \geq 1}$ be the Markov chain defined by the induction formula

$$(s_n, h_n) = \mathbf{T}_{\beta_n}(h_{n-1}) \quad \text{for each } n \geq 1, \quad \text{together with } \omega_n = \tilde{R}[2 \arcsin(h_{n-1}) - \pi] \omega_{n-1}, \quad (24)$$

where $\beta_n \in K$ are independent random variables distributed under μ . The proof of Theorem 4.1 relies upon approximating the particle trajectory $(X_r, \Omega_r)(t)$ starting from (x_0, ω_0) in terms of the following jump process with values in $\mathbf{R}^2 \times \mathbf{S}^1 \times \mathbf{R}_+ \times [-1, 1]$ with the help of Proposition 2.1

$$\begin{aligned} (X_t, \Omega_t, S_t, H_t)(x_0, \omega_0, s_0, h_0) &= (x_0 + t\omega_0, \omega_0, s_0 - t, h_0) && \text{for } 0 \leq t < s_0, \\ (X_t, \Omega_t, S_t, H_t)(x_0, \omega_0, s_0, h_0) &= (X_{\tau_n} + (t - s_n)\omega_n, \omega_n, s_{n+1} - t, h_n) && \text{for } s_n \leq t < s_{n+1}. \end{aligned} \quad (25)$$

Unlike in the case of a random (Poisson) distribution of obstacles, the successive impact parameters on each particle path are not independent and uniformly distributed in the periodic case — likewise, the successive free path lengths on each particle path are not independent with exponential distribution. The Markov chain (24) is introduced to handle precisely this difficulty. Finally, the function $E \equiv E(s, h)$ defined by $E(s, h) = \int_S^\infty \int_{-1}^1 P(\tau, h|h') dh' d\tau$ is a particular (equilibrium) solution of (22).

Note added in proof

After submission of the present paper, we learned that J. Marklof and A. Strömbergsson [9] have proved the limit (16) in any space dimension by a completely different method and without any explicit formula as (17). A later preprint [10] by the same authors proposes a proof of the Markov property for the limiting process without assuming (H).

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