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Differential Geometry

Modular classes of Loday algebroids

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Abstract

We introduce the concept of Loday algebroids, a generalization of Courant algebroids. We define the naive cohomology and modular class of a Loday algebroid, and we show that the modular class of the double of a Lie bialgebroid vanishes. For Courant algebroids, we describe the relation between the naive and standard cohomologies and we conjecture that they are isomorphic when the Courant algebroid is transitive. ***To cite this article: M. Stiénon, P. Xu, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Algébroïdes de Loday. Nous introduisons le concept d'algébroïde de Loday, une généralisation des algébroïdes de Courant, en définissons la cohomologie naïve et la classe modulaire, et nous montrons que la classe modulaire du double d'un bigébroïde de Lie est nulle. Dans le cas des algébroïdes de Courant, nous décrivons la relation entre les cohomologies naïve et standard et nous conjecturons qu'elles sont isomorphes quand l'algébroïde de Courant est transitif. ***Pour citer cet article : M. Stiénon, P. Xu, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Version française abrégée

Les algébroïdes de Courant furent introduits dans [7]. Ils peuvent être définis alternativement à partir du crochet asymétrique de Dorfman [9]. Dans cette Note, nous introduisons le concept d'algébroïde de Loday, une généralisation des algébroïdes de Courant.

Un algébroïde de Loday est un algébroïde de Leibniz, au sens de [4,2,12], dont le fibré vectoriel sous-jacent est équippé d'une pseudo-métrique $\langle\cdot,\cdot\rangle$ et qui jouit de propriétés similaires à celles d'un algébroïde de Courant, à l'exception de la relation $\rho(x)\langle y,z\rangle = \langle x\circ y,z\rangle + \langle y,x\circ z\rangle$.

L'algèbre de Lie–Rinehart sur $C^\infty(M)$ qu'un algébroïde de Leibniz $(E,\rho,\circ)$ induit de façon canonique sur $\Gamma(E)/V$ (où V est le $C^\infty(M)$-module engendré par $\mathcal{D}C^\infty(M)$) permet de construire un complexe de cochaînes

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à partir des sections lisses d'un algébroïde de Loday. La cohomologie de ce complexe est appelée la cohomologie naïve de l'algébroïde de Loday.

Dans [9], Roytenberg définit la cohomologie standard d'un algébroïde de Courant $(E, \rho, [\![\cdot,\cdot]\!], \langle\cdot,\cdot\rangle)$ comme suit. La pseudo-métrique $\langle\cdot,\cdot\rangle$ induit une structure super-Poisson sur la super-variété $E[1]$. De plus, il existe une réalisation symplectique minimale $X \xrightarrow{\pi} E[1]$ et une fonction cubique Θ sur X satisfaisant $\{\Theta,\Theta\} = 0$ telles que, pour tous $f \in C^\infty(M)$ et $e_1, e_2 \in \Gamma(E)$, on a $\mathcal{D}f = \{\Theta, f\}$ et $e_1 \circ e_2 = \{\{\Theta, e_1,\}, e_2\}$. Ici, on conçoit les éléments de $\Gamma(\wedge^k E)$ comme des fonctions de degré k sur X en les considérant comme des fonctions sur $E[1]$ via la pseudo-métrique $\langle\cdot,\cdot\rangle$ et en les identifiant à leur image inverse par π. De façon similaire, les fonctions sur M sont elles aussi identifiées à leur image inverse sur X. Si $\mathcal{A}^k$ désigne l'espace des fonctions de degré k sur X, alors $(\mathcal{A}^\bullet, \{\Theta,\cdot\})$ est un complexe de cochaînes dont la cohomologie $H^\bullet_{\mathrm{std}}(E)$ est nommée *cohomologie standard* par Roytenberg [9].

Théorème 1. *Il existe un homomorphisme naturel* $\phi : H^\bullet_{\mathrm{naive}}(E) \to H^\bullet_{\mathrm{std}}(E)$.

Lorsque E est l'algébroïde de Courant standard $TM \oplus T^*M$, $H^\bullet_{\mathrm{naive}}(E)$ et $H^\bullet_{\mathrm{std}}(E)$ sont isomorphes à la cohomologie de de Rham. Et lorsque E est une algèbre de Lie munie d'une forme bilinéaire non dégénérée ad-invariante, alors $H^\bullet_{\mathrm{naive}}(E)$ et $H^\bullet_{\mathrm{std}}(E)$ sont toutes deux isomorphes à la cohomologie de l'algèbre de Lie. Nous conjecturons que ϕ est un isomorphisme si l'algébroïde de Courant est transitif.

Si un fibré en droites $S \to M$ est muni d'une structure de module sur l'algébroïde de Loday—c'est à dire qu'il existe une application $\mathbb{R}$-linéaire $\Gamma(E) \otimes \Gamma(S) \to \Gamma(S) : e \otimes s \mapsto \nabla_e s$ satisfaisant (13)–(14)—on peut définir sa classe modulaire $[\theta_S] \in H^1_{\mathrm{naive}}(E)$ comme exposé dans [1].

Théorème 2. *Étant donné un algébroïde de Loday* $(E, \rho, \circ, \langle\cdot,\cdot\rangle)$, *le fibré en droites* $\wedge^{top} E$ *est naturellement un module sur* E.

Définition 3. La classe $[\theta] \in H^1_{\mathrm{naive}}(E)$ associée au module $\wedge^{top} E$ est appelée la classe modulaire de l'algébroïde de Loday E.

Dans le cas particulier des doubles d'algébroïdes de Lie, nous avons le résultat suivant :

Théorème 4. *Si un algébroïde de Courant est le double d'un algébroïde de Lie, sa classe modulaire est nulle.*

1. Naive cohomology

Given a Courant algebroid $(E, \rho, [\![\cdot,\cdot]\!], \langle\cdot,\cdot\rangle)$, let $\Gamma(\wedge^k \ker\rho)$ denote the space of smooth sections of the (possibly singular) vector bundle $\wedge^k \ker\rho$ (i.e. smooth sections α of $\wedge^k E$ such that $\alpha|_m \in \wedge^k \ker\rho$ for each $m \in M$). The extension of the pseudo-metric $\langle\cdot,\cdot\rangle$ to $\wedge^k E$ naturally induces an isomorphism $\Xi : \wedge^k E \to \wedge^k E^*$. Since, by definition, $\langle \mathcal{D}f, e\rangle = \frac{1}{2}\rho(e)f$, the sections of $\wedge^k \ker\rho$ are characterized as the elements $\varepsilon \in \Gamma(\wedge^k E)$ such that $\breve{\imath}_{\mathcal{D}f}\varepsilon = 0$, $\forall f \in C^\infty(M)$. Here $\breve{\imath}_{\mathcal{D}f} = \Xi^{-1} \circ i_{\mathcal{D}f} \circ \Xi$, where $i_{\mathcal{D}f} : \Gamma(\wedge^{k+1}E^*) \to \Gamma(\wedge^k E^*)$ is the usual contraction of exterior forms with the section $\mathcal{D}f \in \Gamma(E)$. Define an operator $\breve{d} : \Gamma(\wedge^k \ker\rho) \to \Gamma(\wedge^{k+1}E)$ by

$$(\breve{d}\alpha)(e_0,\ldots,e_k) = \sum_{i=0}^{k} (-1)^i \rho(e_i)\alpha(e_0,\ldots,\widehat{e_i},\ldots,e_k) + \sum_{i<j} (-1)^{i+j} \alpha\big([\![e_i,e_j]\!], e_0,\ldots,\widehat{e_i},\ldots,\widehat{e_j},\ldots,e_k\big),$$

for all $\alpha \in \Gamma(\wedge^k \ker\rho)$ and $e_0,\ldots,e_k \in \Gamma(E)$. Here the pairing between $\Gamma(\wedge^k \ker\rho)$ and $\Gamma(\wedge^k E)$ is via the identification $\Xi : \wedge^k E \to \wedge^k E^*$. The following lemma follows from the Courant algebroid properties, in particular the relations $\rho(\mathcal{D}f) = 0$ and $[\![\mathcal{D}f, e]\!] + \mathcal{D}\langle \mathcal{D}f, e\rangle = 0$:

Lemma 1. *We have* $\breve{d}\Gamma(\wedge^k \ker\rho) \subset \Gamma(\wedge^{k+1}\ker\rho)$. *Moreover,* $(\Gamma(\wedge^\bullet \ker\rho), \breve{d})$ *is a cochain complex.*

The cohomology of this cochain complex is called the naive cohomology of E and is denoted $H^\bullet_{\mathrm{naive}}(E)$.

Remark 1.1. It is easy to see that a 1-cochain $\theta \in \Gamma(\ker\rho)$ is a 1-cocycle if, and only if, $\langle\theta, [\![a,b]\!]\rangle = \rho(a)\langle\theta,b\rangle - \rho(b)\langle\theta,a\rangle$ for all $a,b \in \Gamma(E)$, and a 1-coboundary if, and only if, $\theta = \mathcal{D}f$ for some $f \in C^\infty(M)$.

Remark 1.2. Let V be the $C^\infty(M)$-module generated by $\mathcal{D}(C^\infty(M))$. Since $\langle\mathcal{D}f, a\rangle = \frac{1}{2}\rho(a)f$, we have $V = \Gamma((\ker\rho)^\perp)$ and $\Xi(V) = \Gamma((\ker\rho)^0) = \rho^*(\Gamma(T^*M))$. Moreover $V \subset \Gamma(\ker\rho)$, for $\rho_\circ\mathcal{D} = 0$. Therefore, when $(E, \rho, [\![\cdot,\cdot]\!], \langle\cdot,\cdot\rangle)$ is a regular Courant algebroid (i.e. ρ has constant rank), $E/\Xi^{-1}(\rho^*T^*M)$ is a Lie algebroid and $H^\bullet_{\text{naive}}(E)$ is the cohomology of this Lie algebroid. However, in general, $\Gamma(E)/V$ is only a Lie–Rinehart algebra over $C^\infty(M)$. One can consider $H^\bullet_{\text{naive}}(E)$ as its cohomology [3].

Example 1.3. When $E = TM \oplus T^*M$ is an exact Courant algebroid, $H^\bullet_{\text{naive}}(E)$ is isomorphic to the de Rham cohomology of M.

Example 1.4. If E is a Courant algebroid over a point, i.e. a Lie algebra equipped with a non-degenerate ad-invariant bilinear form, $H^\bullet_{\text{naive}}(E)$ is simply the Lie algebra cohomology.

2. Relation with standard cohomology

Courant algebroids can also be obtained as derived brackets [5,9] using degree two super-symplectic manifolds. More precisely, given a Courant algebroid $(E, \rho, [\![\cdot,\cdot]\!], \langle\cdot,\cdot\rangle)$, $E[1]$ is a super-Poisson manifold, where the Poisson structure is induced by the pseudo-metric. There is a minimal symplectic realization $X \xrightarrow{\pi} E[1]$ and a cubic function Θ on X such that $\{\Theta,\Theta\} = 0$ and, for all $f \in C^\infty(M)$ and $e_1, e_2 \in \Gamma(E)$,

$$\mathcal{D}f = \{\Theta, f\} \quad \text{and} \quad e_1 \circ e_2 = \big\{\{\Theta, e_1\}, e_2\big\},$$

where the symbol $\circ$ denotes the *asymmetric Dorfman bracket* defined by the relation $a \circ b = [\![a,b]\!] + \mathcal{D}\langle a,b\rangle$. Here elements in $\Gamma(\wedge^k E)$ are viewed as functions of degree k on X by considering them as functions on $E[1]$ via the pseudo-metric $\langle\cdot,\cdot\rangle$ and identifying them with their pull back by π. Similarly functions on M are also identified with their pull back in X. By $\mathcal{A}^k$ we denote the space of functions on X of degree k. Then $(\mathcal{A}^\bullet, \{\Theta,\cdot\})$ is a cochain complex. Its cohomology is called the *standard cohomology* by Roytenberg [9] and we shall denote it by $H^\bullet_{\text{std}}(E)$.

Lemma 2. (i) *If* $c \in \Gamma(\wedge^k \ker\rho)$, *then* $\{\Theta, c\} = \breve{d}c$; (ii) *If* $c \in \Gamma(\wedge^k E)$ *satisfies* $\{\Theta, c\} = 0$, *then* $c \in \Gamma(\wedge^k \ker\rho)$ *and* $\breve{d}c = 0$.

Proof. (i) It suffices to prove the case when $k = 1$. The general situation follows from the Leibniz rule. Now since $\rho(c) = 0$, we have $\forall e_1, e_2 \in \Gamma(E)$,

$$\begin{aligned}\langle c \circ e_2, e_1\rangle - (\breve{d}c)(e_1,e_2) &= \big(-\langle e_2 \circ c, e_1\rangle + 2\big\langle\mathcal{D}\langle c, e_2\rangle, e_1\big\rangle\big) - \big(\rho(e_1)\langle c, e_2\rangle - \rho(e_2)\langle c, e_1\rangle - \big\langle c, [\![e_1,e_2]\!]\big\rangle\big)\\ &= \rho(e_2)\langle c, e_1\rangle - \langle e_2 \circ c, e_1\rangle - \big\langle c, [\![e_2,e_1]\!]\big\rangle\\ &= \rho(e_2)\langle c, e_1\rangle - \langle e_2 \circ c, e_1\rangle - \langle c, e_2 \circ e_1\rangle = 0.\end{aligned}$$

It thus follows that $\{\{\{\Theta, c\}, e_2\}, e_1\} - \{\{\breve{d}c, e_2\}, e_1\} = 0$, which implies that $\{\Theta, c\} = \breve{d}c$.

(ii) Since $\breve{\imath}_{\mathcal{D}f}c = \{\mathcal{D}f, c\} = \{\{\Theta, f\}, c\} = \{f, \{\Theta, c\}\} = 0$ for all $f \in C^\infty(M)$, we have $c \in \Gamma(\wedge^k \ker\rho)$. □

As a consequence, we have a homomorphism $\phi : H^\bullet_{\text{naive}}(E) \to H^\bullet_{\text{std}}(E)$. Lemma 2 also implies that ϕ is an isomorphism in degrees 0 and 1. It is natural to ask when ϕ is an isomorphism in all degrees. When E is a Courant algebroid over a point, ϕ is clearly an isomorphism. On the other hand, when E is the standard Courant algebroid $TM \oplus T^*M$, both $H^\bullet_{\text{naive}}(E)$ and $H^\bullet_{\text{std}}(E)$ are isomorphic to the de Rham cohomology of M. Hence ϕ is also an isomorphism. This leads to the following:

Conjecture. *When E is a transitive Courant algebroid, ϕ is an isomorphism.*

3. Lie derivatives and Loday algebroids

The Lie derivative of Courant algebroids was introduced in [10]. Let us recall its definition briefly. An infinitesimal automorphism of the vector bundle $E \xrightarrow{\pi} M$ is a vector field on E—a derivation of the algebra $C^\infty(E)$—which preserves the subspaces $\pi^*C^\infty(M)$ and $\Gamma(E)$ (whose elements are identified with functions linear on the fibers of π through the pairing $\langle\cdot,\cdot\rangle$). In other words, it is a covariant differential operator on E, i.e. a pair of differential operators $\delta^0 : C^\infty(M) \to C^\infty(M)$ and $\delta^1 : \Gamma(E) \to \Gamma(E)$ satisfying

$$\delta^0(fg) = f\delta^0(g) + \delta^0(f)g \quad \text{and} \quad \delta^1(fe) = f\delta^1(e) + \delta^0(f)e,$$

for any $f, g \in C^\infty(M)$ and $e \in \Gamma(E)$. It is known [9] that the Lie algebra $\mathfrak{aut}(E)$ of infinitesimal automorphisms of the Courant algebroid E consists of those covariant differential operators $\delta = (\delta^0, \delta^1)$ on E which satisfy the additional properties:

$$\delta^0\langle e_1, e_2\rangle = \langle \delta^1 e_1, e_2\rangle + \langle e_1, \delta^1 e_2\rangle \quad \text{and} \quad \delta^1[\![e_1, e_2]\!] = [\![\delta^1 e_1, e_2]\!] + [\![e_1, \delta^1 e_2]\!],$$

for all $e_1, e_2 \in \Gamma(E)$.

For any $e \in \Gamma(E)$, the pair $\delta_e = (\delta_e^0, \delta_e^1)$ defined by the relations $\delta_e^0(f) = \rho(e)f$ and $\delta_e^1(x) = e \circ x$ is an infinitesimal automorphism of the Courant algebroid E, i.e. $\delta_e \in \mathfrak{aut}(E)$. Let us denote the (local) flow generated by the vector field on E corresponding to δ_e by ϕ_t. By abuse of notations, we use the same symbol ϕ_t (resp. ϕ_t^*) to denote its induced flow on the tensor bundles $E^i_j = (\otimes^i E) \otimes (\otimes^j E^*)$ $(i, j \in \{0, 1, 2, \ldots\})$ (resp. the induced action on the spaces of sections of the E^i_j's). For any section $\sigma \in \Gamma(E^i_j)$, define the Lie derivative $\mathcal{L}_z\sigma \in \Gamma(E^i_j)$ by $\mathcal{L}_z\sigma = \frac{\mathrm{d}}{\mathrm{d}\tau}\phi_\tau^*\sigma|_{\tau=0}$. Thus we have the usual identity: $\frac{\mathrm{d}}{\mathrm{d}\tau}\phi_\tau^*\sigma|_{\tau=t} = \phi_t^*(\mathcal{L}_z\sigma)$. In the following proposition, we give a list of important properties of this Lie derivative:

Proposition 3. *For all $f, g \in C^\infty(M)$ and $x, y, z \in \Gamma(E)$, we have*:

$$\mathcal{L}_z f = \rho(z)f, \quad \mathcal{L}_z x = z \circ x, \quad \mathcal{L}_{\mathcal{D}f} x = 0, \quad \mathcal{L}_x \mathcal{D}f = \mathcal{D}\mathcal{L}_x f, \tag{1}$$

$$[\delta, \mathcal{L}_z] = \mathcal{L}_{\delta^1 z} \quad \forall \delta \in \mathfrak{aut}(E), \tag{2}$$

$$\mathcal{L}_z(\sigma \otimes \tau) = \mathcal{L}_z\sigma \otimes \tau + \sigma \otimes \mathcal{L}_z\tau \quad \forall \sigma, \tau \in \oplus_{i,j} E^i_j, \tag{3}$$

$$\mathcal{L}_{[\![x,y]\!]} = [\mathcal{L}_x, \mathcal{L}_y], \tag{4}$$

$$\mathcal{L}_z[\![x, y]\!] = [\![\mathcal{L}_z x, y]\!] + [\![x, \mathcal{L}_z y]\!], \tag{5}$$

$$\mathcal{L}_{fx} y = f\mathcal{L}_x y - \big(\rho(y)f\big)x + 2\langle x, y\rangle\mathcal{D}f, \tag{6}$$

$$\mathcal{L}_z\langle x, y\rangle = \langle \mathcal{L}_z x, y\rangle + \langle x, \mathcal{L}_z y\rangle. \tag{7}$$

Definition 4. A Loday algebroid consists of a vector bundle $\pi : E \to M$, a pseudo-metric $\langle\cdot,\cdot\rangle$ on the fibers of π, a bundle map $\rho : E \to TM$ and an $\mathbb{R}$-bilinear operation $\circ$ on $\Gamma(E)$ satisfying

$$e_1 \circ (e_2 \circ e_3) = (e_1 \circ e_2) \circ e_3 + e_2 \circ (e_1 \circ e_3), \tag{8}$$

$$\rho(e_1 \circ e_2) = \big[\rho(e_1), \rho(e_2)\big], \tag{9}$$

$$e_1 \circ (fe_2) = \big(\rho(e_1)f\big)e_2 + f(e_1 \circ e_2), \tag{10}$$

$$e_1 \circ e_2 + e_2 \circ e_1 = 2\mathcal{D}\langle e_1, e_2\rangle, \tag{11}$$

$$\mathcal{D}f \circ e = 0, \tag{12}$$

where $\mathcal{D} : C^\infty(M) \to \Gamma(E)$ is the $\mathbb{R}$-linear map defined by $\langle \mathcal{D}f, e\rangle = \frac{1}{2}\rho(e)f$.

Remark 3.1. (i) According to [11], for Courant algebroids, axioms (9) and (10) are redundant. It would be interesting to investigate if it is also the case for Loday algebroids.

(ii) Leibniz algebroids – so called because their space of sections is a Leibniz algebra in the sense of [8] – are a more general notion that was studied by several authors [4,2,12].

(iii) A Courant algebroid is a Loday algebroid satisfying the additional axiom $\rho(e)\langle e_1, e_2\rangle = \langle e \circ e_1, e_2\rangle + \langle e_1, e \circ e_2\rangle$.

Lemma 5. *If* $(E, \rho, \circ, \langle\cdot,\cdot\rangle)$ *is a Loday algebroid, then* $\rho(\mathcal{D}f) = 0$ *and* $[\![\mathcal{D}f, e]\!] + \mathcal{D}\langle\mathcal{D}f, e\rangle = 0$*, for all* $f \in C^\infty(M)$ *and* $e \in \Gamma(E)$*. Here* $[\![x, y]\!] = \frac{1}{2}(x \circ y - y \circ x)$ *as in a Courant algebroid.*

Proof. Applying ρ to both sides of (11) and making use of (9), we get $\rho(\mathcal{D}\langle e_1, e_2\rangle) = 0$ for any $e_1, e_2 \in \Gamma(E)$ and thus also $\rho(\mathcal{D}\langle fe_1, e_2\rangle) = 0$ for any $f \in C^\infty(M)$. The Leibniz rule $\mathcal{D}(fg) = g\mathcal{D}f + f\mathcal{D}g$ implies that $\rho(\mathcal{D}f) = 0$. The other relation follows immediately from (12) and (11). □

As a consequence, the definition of the naive cohomology extends from Courant algebroids to Loday algebroids.

Let $(E, \rho, \circ, \langle\cdot,\cdot\rangle)$ be a Loday algebroid. Given a section $z \in \Gamma(E)$, set $\mathcal{L}_z f = \rho(z)f$ for $f \in C^\infty(M)$ and $\mathcal{L}_z x = z \circ x$ for $x \in \Gamma(E)$ and extend $\mathcal{L}_z$ to $\Gamma(\wedge^k E)$ by the Leibniz rule.

Proposition 6. *Identities* (4)*,* (5) *and* (6) *still hold for any Loday algebroid.*

Remark 3.2. It is unknown if the standard cohomology can be defined for Loday algebroids. Indeed, it would be interesting to see if there exists a derived bracket in the sense of Kosmann–Schwarzbach [5] for a Loday algebroid.

4. Modular classes

A Loday algebroid module is a vector bundle $S \to M$ endowed with an $\mathbb{R}$-linear map $\Gamma(E) \otimes \Gamma(S) \to \Gamma(S) : e \otimes s \mapsto \nabla_e s$ satisfying

$$\nabla_{\mathcal{D}f} s = 0, \qquad \nabla_e(fs) = f\nabla_e s + \big(\rho(e)f\big)s, \tag{13}$$

$$\nabla_{fe} s = f\nabla_e s, \qquad \nabla_{e_1}(\nabla_{e_2} s) - \nabla_{e_2}(\nabla_{e_1} s) = \nabla_{[\![e_1, e_2]\!]} s \tag{14}$$

for any $f \in C^\infty(M)$, $e, e_1, e_2 \in \Gamma(E)$ and $s \in \Gamma(S)$.

Now let S be a real line bundle which is a module of the Loday algebroid E. Assume that there exists a nowhere zero section $s \in \Gamma(S)$. The relation $D_e s = \langle\theta_s, e\rangle s$ defines a section $\theta_s \in \Gamma(E)$. From $\nabla_{\mathcal{D}f} s = 0$, it follows that $\rho(\theta_s) = 0$. And from (14), it follows that θ_s is a naive 1-cocycle. Finally, (13) implies that, for any nowhere vanishing function $f \in C^\infty(M)$, $\theta_{fs} = f\theta_s + 2\mathcal{D}(\ln|f|)$. Thus the class $[\theta_s] \in H^1_{\text{naive}}(E)$ is independent of the chosen section s and only depends on the module S. We will denote this class by θ_S. As in [1,6], when the line bundle S is not trivial, we set $\theta_S = \frac{1}{2}\theta_{S\otimes S}$, where $S \otimes S$ is necessarily a trivial real line bundle. We call θ_S the modular class of the module S.

Theorem 7. *Given a Loday algebroid* $(E, \rho, \circ, \langle\cdot,\cdot\rangle)$*,* $\wedge^{top} E$ *is an* E*-module with* $\nabla = \mathcal{L}$*.*

Proof. It remains to prove that $\mathcal{L}_{fe} s = f\mathcal{L}_e s$ for any $f \in C^\infty(M)$ and $s \in \Gamma(\wedge^{top} E)$. According to Proposition 3, for any $f \in C^\infty(M)$ and $e, a \in \Gamma(E)$, we have

$$\begin{aligned}\mathcal{L}_{fe} a &= f\mathcal{L}_e a - \big(\rho(a)f\big)e + 2\langle e, a\rangle\mathcal{D}f = f\mathcal{L}_e a - 2\langle\mathcal{D}f, a\rangle e + 2\langle e, a\rangle\mathcal{D}f \\ &= \big(f\mathcal{L}_e - 2(e\wedge)_\circ\check{\imath}_{\mathcal{D}f} + 2(\mathcal{D}f\wedge)_\circ\check{\imath}_e\big)(a).\end{aligned}$$

Note that, as differential operators on $\Gamma(\wedge^\bullet E)$, $\mathcal{L}_{fe}$, $f\mathcal{L}_e$, $2(e\wedge)_\circ\check{\imath}_{\mathcal{D}f}$ and $2(Df\wedge)_\circ\check{\imath}_e$ all are derivations of degree 0 with respect to the wedge product on $\Gamma(\wedge^\bullet E)$. Since $\mathcal{L}_{fe}$ and $f\mathcal{L}_e - 2(e\wedge)_\circ\check{\imath}_{\mathcal{D}f} + 2(Df\wedge)_\circ\check{\imath}_e$ are equal when acting both on sections of E and on functions on M, they are also equal when extended to $\Gamma(\wedge^* E)$. In particular, if $s \in \Gamma(\wedge^{top} E)$,

$$\mathcal{L}_{fe} s = f\mathcal{L}_e s - 2(e\wedge)_\circ\check{\imath}_{\mathcal{D}f} s + 2(\mathcal{D}f\wedge)_\circ\check{\imath}_e s = f\mathcal{L}_e s - 2\langle e, \mathcal{D}f\rangle s + 2\langle\mathcal{D}f, e\rangle s = f\mathcal{L}_e s. \qquad \square$$

The modular class $[\theta_{\wedge^{top} E}] \in H^1_{\text{naive}}(E)$ of the E-module $\wedge^{top} E$ is called the *modular class of the Loday algebroid* E.

5. Examples

Let $E = A \oplus A^*$ be the double of a Lie bialgebroid (A, A^*) [7]. In this case, $\mathcal{D} = \frac{1}{2}(d + d_*)$ and, for all $X, Y \in \Gamma(A)$ and $\xi, \eta \in \Gamma(A^*)$, the bracket on $\Gamma(E)$ is defined by

$$[\![X, \xi]\!] = \big(-L_\xi X + \tfrac{1}{2} d_*(\xi, X)\big) + \big(L_X \xi - \tfrac{1}{2} d(\xi, X)\big), \qquad [\![X, Y]\!] = [X, Y], \qquad [\![\xi, \eta]\!] = [\xi, \eta].$$

Now $\wedge^{top} E \cong (\wedge^{top} A) \otimes (\wedge^{top} A^*)$ is a trivial line bundle. For the sake of simplicity, we assume that there exists a nowhere vanishing section $V \in \Gamma(\wedge^{top} A)$. Let $\Omega \in \Gamma(\wedge^{top} A^*)$ be its dual section. For any $X \in \Gamma(A)$ and $\xi \in \Gamma(A^*)$, one has $\mathcal{L}_X \xi = [\![X, \xi]\!] + \mathcal{D}\langle X, \xi\rangle = -L_\xi X + d_*(\xi, X) + L_X \xi = -i_\xi d_* X + L_X \xi$. It follows from the Leibniz rule (see Proposition 3) that, for any $\sigma \in \Gamma(\wedge^k A^*)$, $\mathcal{L}_X \sigma = L_X \sigma + \lambda$ where $\lambda \in \Gamma(A \otimes (\wedge^{k-1} A^*))$. On the other hand, since A is isotropic with respect to $\langle \cdot, \cdot \rangle$, we have that $\mathcal{L}_X \tau = L_X \tau$ if $\tau \in \Gamma(\wedge^k A)$. It thus follows that

$$\mathcal{L}_X (V \wedge \Omega) = (\mathcal{L}_X V) \wedge \Omega + V \wedge (\mathcal{L}_X \Omega) = (L_X V) \wedge \Omega + V \wedge (L_X \Omega) = 0.$$

Similarly we have $\mathcal{L}_\xi (V \wedge \Omega) = 0$, for all $\xi \in \Gamma(A^*)$. Thus we have proved:

Theorem 8. *If a Courant algebroid is the double of a Lie bialgebroid, then its modular class vanishes.*

Example 5.1. If $E = TM \oplus T^*M$ is an exact Courant algebroid, the Courant bracket is given by

$$[\![X + \xi, Y + \eta]\!] = [X, Y] + i_{X \wedge Y} \phi + L_X \eta - L_Y \xi + \tfrac{1}{2} d\big((\xi, Y) - (\eta, X)\big),$$

for all $X, Y \in \mathfrak{X}(M)$ and $\xi, \eta \in \Omega^1(M)$. Here ϕ is a closed 3-form.

Take a nowhere zero $V \in \Gamma(\wedge^{top} TM)$ and its dual $\Omega \in \Omega^{top}(M)$. One easily sees that $\mathcal{L}_X V = L_X V + V'$, where $V' \in \Gamma(T^*M \otimes \wedge^{top-1} TM)$ and $\mathcal{L}_X \Omega = L_X \Omega$. Thus it follows that $\mathcal{L}_X (V \wedge \Omega) = L_X (V \wedge \Omega) = 0$, $\forall X \in \Gamma(TM)$. One also sees that $\mathcal{L}_\xi (V \wedge \Omega) = 0$, $\forall \xi \in \Gamma(T^*M)$. Therefore the modular class vanishes.

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