

Partial Differential Equations

# Approximation of diffuse measures for parabolic capacities

Francesco Petitta<sup>a</sup>, Augusto C. Ponce<sup>b</sup>, Alessio Porretta<sup>c</sup>

<sup>a</sup> Dipartimento di Matematica, Università di Roma La Sapienza, Piazzale A. Moro 2, 00185 Roma, Italy

<sup>b</sup> Laboratoire de mathématiques et physique théorique (CNRS UMR 6083), Université de Tours, 37200 Tours, France

<sup>c</sup> Dipartimento di Matematica, Università di Roma Tor Vergata, Via della Ricerca Scientifica 1, 00133 Roma, Italy

Received and accepted 25 November 2007

Available online 14 January 2008

Presented by Haïm Brezis

## Abstract

Given a parabolic cylinder  $Q = (0, T) \times \Omega$ , with  $\Omega \subset \mathbb{R}^N$ , we consider the class of finite measures which do not charge sets of zero  $p$ -parabolic capacity in  $Q$ . We prove that such measures can be strongly approximated by measures which can be written as  $v_t - \Delta_p v$  with  $v \in L^p(0, T; W_0^{1,p}(\Omega)) \cap L^\infty(Q)$ . Estimates on the capacity of level sets of solutions of parabolic equations play a crucial role in our proof. **To cite this article:** F. Petitta et al., C. R. Acad. Sci. Paris, Ser. I 346 (2008).

© 2007 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

## Résumé

**Approximation des mesures diffuses pour des capacités paraboliques.** Étant donné un cylindre parabolique  $Q = (0, T) \times \Omega$ , avec  $\Omega \subset \mathbb{R}^N$ , on considère la classe des mesures bornées sur  $Q$  qui ne chargent pas les ensembles de  $p$ -capacité nulle. Nous démontrons que ces mesures peuvent être approchées au sens fort par des mesures de la forme  $v_t - \Delta_p v$  avec  $v \in L^p(0, T; W_0^{1,p}(\Omega)) \cap L^\infty(Q)$ . Des estimations sur la capacité des ensembles de niveau des solutions d'équations paraboliques jouent un rôle crucial dans notre preuve. **Pour citer cet article :** F. Petitta et al., C. R. Acad. Sci. Paris, Ser. I 346 (2008).

© 2007 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

## Version française abrégée

Étant donné un ouvert borné  $\Omega \subset \mathbb{R}^N$  et  $T > 0$ , on pose  $Q = (0, T) \times \Omega$ . Nous rappelons que pour tout  $p > 1$  et pour tout ouvert  $U \Subset Q$ , la  $p$ -capacité parabolique de  $U$  est définie par (voir [6,3])

$$\text{cap}_p(U) = \inf \{ \|u\|_W : u \in W, u \geq \chi_U \text{ p.p. dans } Q \}, \quad (1)$$

où  $W = \{u \in L^p(0, T; V) : u_t \in L^{p'}(0, T; V')\}$ ,  $V = W_0^{1,p}(\Omega) \cap L^2(\Omega)$  et  $V'$  est son dual. On considère  $W$  muni de sa norme usuelle (5). La définition de capacité parabolique  $\text{cap}_p$  est étendue aux sous-ensembles boréliens de  $Q$  de façon usuelle.

E-mail addresses: [petitta@mat.uniroma1.it](mailto:petitta@mat.uniroma1.it) (F. Petitta), [ponce@lmpt.univ-tours.fr](mailto:ponce@lmpt.univ-tours.fr) (A.C. Ponce), [porretta@mat.uniroma2.it](mailto:porretta@mat.uniroma2.it) (A. Porretta).

Dans la suite on désigne par  $\mathcal{M}(Q)$  l'espace des mesures de Radon bornées dans  $Q$  avec la norme  $\|\mu\|_{\mathcal{M}(Q)} = |\mu|(Q)$ . On dira qu'une mesure bornée  $\mu$  est *diffuse* si  $\mu(E) = 0$  pour tous les boréliens  $E \subset Q$  tels que  $\text{cap}_p(E) = 0$ . On désignera par  $\mathcal{M}_0(Q)$  le sous-espace des mesures diffuses dans  $Q$ .

Les mesures diffuses jouent un rôle important dans l'étude des équations aux dérivées partielles avec données singulières. On peut souvent démontrer pour ces mesures des résultats d'existence et d'unicité analogues à ceux dans le cadre variationnel. Dans ce but, plusieurs propriétés des mesures diffuses ont été établies dans [3]; les auteurs y montrent en particulier que pour tout  $\mu \in \mathcal{M}_0(Q)$  ils existent  $f \in L^1(Q)$ ,  $g \in L^p(0, T; V)$  et  $h \in L^{p'}(0, T; W^{-1,p'}(\Omega))$  tels que

$$\mu = f + g_t + h \quad \text{dans } \mathcal{D}'(Q). \quad (2)$$

On ne sait pas si toute mesure qui admet une telle décomposition est diffuse. Dans [5], nous démontrons que ceci est le cas si  $\mu$  vérifie (2) avec  $g$  dans  $L^\infty(Q)$  (voir Proposition 1.1 ci-dessous).

Sachant que la décomposition (2) n'est pas unique, on peut se demander si toute mesure diffuse admet une décomposition de la forme (2) avec  $g \in L^\infty(Q)$ . En fait, la réponse est négative (voir l'Exemple 1 ci-dessous). On démontre cependant dans cette Note que tout  $\mu \in \mathcal{M}_0(Q)$  peut être approché au sens fort par des mesures qui ont cette propriété. Plus précisément,

**Théorème 0.1.** *Soit  $\mu \in \mathcal{M}_0(Q)$ . Pour tout  $\varepsilon > 0$  il existe  $\nu \in \mathcal{M}_0(Q)$  tel que*

$$\|\mu - \nu\|_{\mathcal{M}(Q)} \leq \varepsilon \quad \text{et} \quad \nu = \nu_t - \Delta_p \nu \quad \text{dans } \mathcal{D}'(Q), \quad (3)$$

avec  $\nu \in L^p(0, T; W_0^{1,p}(\Omega)) \cap L^\infty(Q)$ .

Cette propriété de densité a plusieurs applications dans l'étude des problèmes paraboliques non linéaires. Par exemple, le Théorème 0.1 nous permet de démontrer l'existence des solutions de l'équation (8) pour toutes données  $\mu \in \mathcal{M}_0(\Omega)$ ,  $u_0 \in L^1(\Omega)$  et pour toute fonction croissante et continue  $f : \mathbb{R} \rightarrow \mathbb{R}$  avec  $f(0) = 0$ . Ce résultat, ainsi que d'autres conséquences du Théorème 0.1, sera présenté dans [5]. Cette Note est consacrée à la démonstration du Théorème 0.1, qui d'ailleurs repose sur des résultats préliminaires qui ont leur propre intérêt. Parmi eux, nous démontrons une estimation a priori sur la capacité des ensembles de niveau pour les solutions de l'équation du  $p$ -laplacien parabolique (voir Proposition 1.3 ci-dessous). Cette estimation semble nouvelle dans le contexte des équations paraboliques non linéaires.

## 1. Introduction and main results

Given a bounded open set  $\Omega \subset \mathbb{R}^N$  and  $T > 0$ , let  $Q = (0, T) \times \Omega$ . We recall that for every  $p > 1$  and every open subset  $U \Subset Q$ , the *parabolic  $p$ -capacity* of  $U$  is given by (see [6,3])

$$\text{cap}_p(U) = \inf\{\|u\|_W : u \in W, u \geq \chi_U \text{ a.e. in } Q\}, \quad (4)$$

where  $W = \{u \in L^p(0, T; V) : u_t \in L^{p'}(0, T; V')\}$ ,  $V = W_0^{1,p}(\Omega) \cap L^2(\Omega)$  and  $V'$  is its dual space;  $W$  is endowed with the norm

$$\|u\|_W = \|u\|_{L^p(0,T;V)} + \|u_t\|_{L^{p'}(0,T;V')}. \quad (5)$$

The parabolic capacity  $\text{cap}_p$  is then extended to arbitrary Borel subsets of  $Q$  in a standard way.

We denote by  $\mathcal{M}(Q)$  the set of all bounded Radon measures in  $Q$  equipped with the norm  $\|\mu\|_{\mathcal{M}(Q)} = |\mu|(Q)$ . We call a bounded measure  $\mu$  *diffuse* if  $\mu(E) = 0$  for every Borel set  $E \subset Q$  such that  $\text{cap}_p(E) = 0$ ;  $\mathcal{M}_0(Q)$  will denote the subspace of all diffuse measures in  $Q$ .

Diffuse measures play an important role in the study of boundary value problems with measures as source terms. Indeed, for such measures one expects to obtain counterparts (in some generalized framework) of existence and uniqueness results known in the variational setting. Properties of diffuse measures in connection with the resolution of nonlinear parabolic problems have been investigated in [3]. In that paper, the authors proved that for every  $\mu \in \mathcal{M}_0(Q)$  there exist  $f \in L^1(Q)$ ,  $g \in L^p(0, T; V)$  and  $h \in L^{p'}(0, T; W^{-1,p'}(\Omega))$  such that

$$\mu = f + g_t + h \quad \text{in } \mathcal{D}'(Q). \quad (6)$$

It is not known whether every measure which can be decomposed in this form is diffuse. However, in [5] we prove the following:

**Proposition 1.1.** *Assume that  $\mu \in \mathcal{M}(\Omega)$  satisfies (6), where  $f \in L^1(Q)$ ,  $g \in L^p(0, T; W_0^{1,p}(\Omega))$  and  $h \in L^{p'}(0, T; W^{-1,p'}(\Omega))$ . If  $g \in L^\infty(Q)$ , then  $\mu$  is diffuse.*

Note that the decomposition in (6) is not uniquely determined. In view of Proposition 1.1 above, a natural question would be whether every diffuse measure can be written as (6) for some  $g \in L^\infty(Q)$ . It turns out that the answer is *no*; see Example 1 below. Our goal in this Note is to show that every  $\mu \in \mathcal{M}_0(Q)$  can be strongly approximated by measures which do have such property. In fact, we establish the following:

**Theorem 1.2.** *Let  $\mu \in \mathcal{M}_0(Q)$ . Then, for every  $\varepsilon > 0$  there exists  $v \in \mathcal{M}_0(Q)$  such that*

$$\|\mu - v\|_{\mathcal{M}(Q)} \leq \varepsilon \quad \text{and} \quad v = v_t - \Delta_p v \quad \text{in } \mathcal{D}'(Q), \tag{7}$$

where  $v \in L^p(0, T; W_0^{1,p}(\Omega)) \cap L^\infty(Q)$ .

In view of Example 1 below, Theorem 1.2 is the best one can expect for an arbitrary diffuse measure  $\mu$ . This density property is nevertheless useful in the study of nonlinear parabolic problems. For instance, using Theorem 1.2 we can prove that for every  $\mu \in \mathcal{M}_0(\Omega)$  and every nondecreasing continuous function  $f : \mathbb{R} \rightarrow \mathbb{R}$  with  $f(0) = 0$ , the problem

$$\begin{cases} u_t - \Delta_p u + f(u) = \mu & \text{in } (0, T) \times \Omega, \\ u = u_0 & \text{on } \{0\} \times \Omega, \\ u = 0 & \text{on } (0, T) \times \partial\Omega, \end{cases} \tag{8}$$

admits a solution for any  $u_0 \in L^1(\Omega)$  (see [5]). Such a result, well-known in the elliptic case, was surprisingly missing for parabolic equations.

This Note will be devoted to the proof of Theorem 1.2. In turn, this proof relies on some results which are interesting in their own, as for the following capacity estimate:

**Proposition 1.3.** *Given  $\mu \in \mathcal{M}(Q) \cap L^{p'}(0, T; W^{-1,p'}(\Omega))$  and  $u_0 \in L^2(\Omega)$ , let  $u \in L^p(0, T; V)$  be the solution of*

$$\begin{cases} u_t - \Delta_p u = \mu & \text{in } (0, T) \times \Omega, \\ u = u_0 & \text{on } \{0\} \times \Omega, \\ u = 0 & \text{on } (0, T) \times \partial\Omega. \end{cases} \tag{9}$$

Then,

$$\text{cap}_p(\{|u| > k\}) \leq C \max \left\{ \frac{1}{k^{(p-1)/p'}}, \frac{1}{k^{(p-1)/p}} \right\} \quad \forall k > 0, \tag{10}$$

where  $C > 0$  is a constant depending on  $\|\mu\|_{\mathcal{M}(Q)}$ ,  $\|u_0\|_{L^1(\Omega)}$ ,  $N$ ,  $p$  and  $\Omega$ .

In (10) we have identified  $u$  with its cap-quasicontinuous representative, which exists since  $u \in W$  (see [3]). In particular, the quantity  $\text{cap}_p(\{|u| > k\})$  is well-defined.

The above estimate on the capacity of level sets seems new in the context of nonlinear parabolic equations. In the next section, we give the proof of Proposition 1.3 in case  $\mu \geq 0$ ,  $u_0 \geq 0$ : the complete argument for signed data will be presented in the forthcoming paper [5].

## 2. Proofs of the main results

**Proof of Proposition 1.3.** To simplify the exposition, we will give the proof in the case  $p \geq \frac{2N}{N+2}$  (this implies that in (5) one simply has  $V = W_0^{1,p}(\Omega)$ ). We shall assume in addition that  $\mu \geq 0$  and  $u_0 \geq 0$ ; hence,  $u \geq 0$ . We define, for any positive  $k$ ,

$$T_k(s) = \max\{-k, \min\{k, s\}\} \quad \text{and} \quad \Psi_k(s) = \frac{T_k(s)}{k} \quad \forall s \in \mathbb{R}.$$

Step 1. Estimates of  $\Psi_k(u)$  in the space  $\Lambda_p = L^\infty(0, T; L^2(\Omega)) \cap L^p(0, T; W_0^{1,p}(\Omega))$ .

Let us multiply the equation in (9) by  $T_k(u)$  and integrate between 0 and  $r$ . We have

$$\int_{\Omega} \Theta_k(u)(r) \, dx + \int_0^r \int_{\Omega} |\nabla T_k(u)|^p \, dx \, dt \leq k \|\mu\|_{\mathcal{M}(\mathcal{Q})} + \int_{\Omega} \Theta_k(u_0) \, dx,$$

where  $\Theta_k(s) = \int_0^s T_k(\sigma) \, d\sigma$ . Observing that  $\frac{T_k(s)^2}{2} \leq \Theta_k(s) \leq ks, \forall s \geq 0$ , we have

$$\frac{1}{k} \int_{\Omega} \frac{T_k(u)^2}{2}(r) \, dx + \frac{1}{k} \int_0^r \int_{\Omega} |\nabla T_k(u)|^p \, dx \, dt \leq \|\mu\|_{\mathcal{M}(\mathcal{Q})} + \|u_0\|_{L^1(\Omega)}$$

for any  $r \in (0, T)$ . Hence,

$$\|\Psi_k(u)\|_{L^\infty(0,T;L^2(\Omega))}^2 \leq \frac{C}{k} \quad \text{and} \quad \|\Psi_k(u)\|_{L^p(0,T;W_0^{1,p}(\Omega))}^p \leq \frac{C}{k^{p-1}}. \tag{11}$$

Step 2. Estimates in  $W$ .

By standard results (see [4]) there exists a unique solution  $z \in L^\infty(0, T; L^2(\Omega)) \cap L^p(0, T; W_0^{1,p}(\Omega))$  of the backward problem

$$\begin{cases} -z_t - \Delta_p z = -2\Delta_p \Psi_k(u) & \text{in } (0, T) \times \Omega, \\ z = \Psi_k(u)(T) & \text{on } \{T\} \times \Omega, \\ z = 0 & \text{on } (0, T) \times \partial\Omega. \end{cases} \tag{12}$$

Let us multiply (12) by  $z$  and integrate between  $r$  and  $T$ . Using Young’s inequality we obtain

$$\int_{\Omega} z^2(r) \, dx + \|z\|_{L^p(r,T;W_0^{1,p}(\Omega))}^p \leq \int_{\Omega} [\Psi_k(u)(T)]^2 \, dx + C \|\Psi_k(u)\|_{L^p(0,T;W_0^{1,p}(\Omega))}^p$$

for every  $r \in [0, T)$ . The equation in (12) implies

$$\|z_t\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))}^{p'} \leq C (\|z\|_{L^p(0,T;W_0^{1,p}(\Omega))}^p + \|\Psi_k(u)\|_{L^p(0,T;W_0^{1,p}(\Omega))}^p).$$

Hence,

$$\begin{aligned} \|z\|_W &\leq C_1 \max \left\{ \|\Psi_k(u)\|_{L^\infty(0,T;L^2(\Omega))}^{2/p}, \|\Psi_k(u)\|_{L^\infty(0,T;L^2(\Omega))}^{2/p'} \right\} \\ &\quad + C_2 \max \left\{ \|\Psi_k(u)\|_{L^p(0,T;W_0^{1,p}(\Omega))}, \|\Psi_k(u)\|_{L^p(0,T;W_0^{1,p}(\Omega))}^{p-1} \right\}, \end{aligned}$$

which yields, using (11),

$$\|z\|_W \leq C \max \left\{ \frac{1}{k^{(p-1)/p'}}, \frac{1}{k^{(p-1)/p}} \right\}. \tag{13}$$

Step 3. Proof completed.

Since  $\mu \geq 0$ , we have  $u_t - \Delta_p u \geq 0$  in  $\mathcal{Q}$ . Thus, by a nonlinear version of Kato’s inequality for parabolic equations (see [5]),

$$\Psi_k(u)_t - \Delta_p \Psi_k(u) \geq 0. \tag{14}$$

Combining (12) and (14) we obtain

$$-z_t - \Delta_p z \geq -\Psi_k(u)_t - \Delta_p \Psi_k(u). \tag{15}$$

Since both  $z$  and  $\Psi_k(u)$  belong to  $L^p(0, T; W_0^{1,p}(\Omega))$ , a standard comparison argument (multiply both sides of (15) by  $(z - \Psi_k(u))^-$ ) allows us to conclude that  $z \geq \Psi_k(u)$  a.e. in  $\mathcal{Q}$ . In particular,  $z \geq 1$  a.e. on  $\{u > k\}$ . On the other hand, since  $u$  belongs to  $W$ , it has a unique cap-quasicontinuous representative (still denoted by  $u$ ), hence the set  $\{u > k\}$  is cap-quasi open and its capacity can be computed as in (4) (see [3, Proposition 2.19]). Therefore,  $\text{cap}_p(\{u > k\}) \leq \|z\|_W$ . Using (13) we obtain (10).  $\square$

Before establishing Theorem 1.2, we will need an important property enjoyed by the convolution of diffuse measures. We first recall the following (see [2]):

**Definition 2.1.** A sequence of measures  $(\mu_n)$  in  $Q$  is  $p$ -equidiffuse if  $(\mu_n)$  is bounded in  $\mathcal{M}(Q)$  and for every  $\varepsilon > 0$  there exists  $\eta > 0$  such that  $\text{cap}_p(E) < \eta \Rightarrow |\mu_n|(E) < \varepsilon \forall n \geq 1$ .

**Proposition 2.2.** Let  $\mu \in \mathcal{M}_0(Q)$  have compact support and let  $(\rho_n)$  be a sequence of mollifiers. Then,  $(\rho_n * \mu)$  is  $p$ -equidiffuse.

**Proof.** It suffices to establish the result when  $\mu \geq 0$  and  $\text{supp } \rho_n \subset B_{r_n}(0)$  for some sequence  $r_n \rightarrow 0$ . Clearly,  $(\rho_n * \mu)$  is bounded in  $\mathcal{M}(Q)$ . Assume by contradiction that  $(\rho_n * \mu)$  is not  $p$ -equidiffuse. By outer regularity of  $\text{cap}_p$ , there exist  $\varepsilon_0 > 0$ , a sequence of open sets  $(\omega_n)$  and  $Q_0 \Subset Q$  such that  $\omega_n \Subset Q_0$ ,  $\text{cap}_p(\omega_n) \rightarrow 0$  and  $\int_{\omega_n} \rho_n * \mu \geq \varepsilon_0 \forall n \geq 1$ . Fix  $Q_1 \Subset Q$  with  $Q_0 \Subset Q_1$ . We claim that there exists a sequence  $(\zeta_n) \subset W$  such that

$$\zeta_n \geq 1 \text{ on } \omega_n, \quad \text{supp } \zeta_n \Subset Q_1 \quad \text{and} \quad \|\check{\rho}_n * \zeta_n\|_W \rightarrow 0. \tag{16}$$

Indeed, take  $\varphi \in C_0^\infty(Q_1)$  such that  $0 \leq \varphi \leq 1$  in  $Q$  and  $\varphi = 1$  in  $Q_0$ . For each  $n \geq 1$ , let  $\tilde{\zeta}_n \in W$  be such that  $\tilde{\zeta}_n \geq \chi_{\omega_n}$  and  $\|\tilde{\zeta}_n\|_W \leq 2 \text{cap}_p(\omega_n)$ . The sequence  $(\zeta_n)$  given by  $\zeta_n = \tilde{\zeta}_n \varphi$  then satisfies (16). Passing to a subsequence if necessary, we may also assume that

$$\check{\rho}_n * \zeta_n(x) \rightarrow 0 \quad \forall x \in Q \setminus E_0, \tag{17}$$

for some Borel set  $E_0 \subset Q$  such that  $\text{cap}_p(E_0) = 0$ . On the other hand, by Jensen’s inequality,

$$0 \leq \check{\rho}_n * T_1(\zeta_n) \leq T_1(\check{\rho}_n * \zeta_n).$$

We thus have for  $n \geq 1$  large enough

$$\varepsilon_0 \leq \int_{\omega_n} \rho_n * \mu \leq \int_Q T_1(\zeta_n) \rho_n * \mu = \int_Q \check{\rho}_n * T_1(\zeta_n) \, d\mu \leq \int_Q T_1(\check{\rho}_n * \zeta_n) \, d\mu.$$

The right-hand side vanishes as  $n \rightarrow \infty$  by (17) and dominated convergence. This is a contradiction.  $\square$

**Proof of Theorem 1.2.** It is enough to consider  $\mu$  with compact support in  $Q$  (since any measure is strongly approximated in  $\mathcal{M}(Q)$  by measures with compact support). Let  $\mu_n = \rho_n * \mu$ , where  $(\rho_n)$  is a sequence of mollifiers. We denote by  $u_n$  the solution of (9) with data  $\mu_n$  and  $u_0 = 0$ . Consider  $k > 0$  to be chosen below. For  $\delta > 0$  small, take the continuous function  $S_{k,\delta}$  given by  $S_{k,\delta}(s) = 1$  if  $|s| \leq k$ ,  $S_{k,\delta}(s) = 0$  if  $|s| > k + \delta$ , and  $S_{k,\delta}$  is affine linear otherwise. Let

$$T_{k,\delta}(s) = \int_0^s S_{k,\delta}(\sigma) \, d\sigma \quad \forall s \in \mathbb{R}.$$

Notice that  $T_{k,\delta}$  converges pointwise to  $T_k$  as  $\delta \rightarrow 0$ .

Given  $\varphi \in C_0^\infty(Q)$ , multiply the equation solved by  $u_n$  by  $S_{k,\delta}(u_n)\varphi$ . We then have

$$T_{k,\delta}(u_n)_t - \text{div}(S_{k,\delta}(u_n)|\nabla u_n|^{p-2}\nabla u_n) = S_{k,\delta}(u_n)\mu_n + \frac{1}{\delta}|\nabla u_n|^p \text{sign}(u_n)\chi_{\{k \leq |u_n| < k+\delta\}} \quad \text{in } \mathcal{D}'(Q). \tag{18}$$

Using  $(1 - S_{k,\delta}(u_n)) \text{sign}(u_n)$  as test function in Eq. (9) for  $u_n$  we obtain

$$\frac{1}{\delta} \int_{\{k \leq |u_n| < k+\delta\}} |\nabla u_n|^p \leq \int_Q |1 - S_{k,\delta}(u_n)| |\mu_n| \leq \int_{\{|u_n| > k\}} |\mu_n|. \tag{19}$$

Thus, the right-hand side of (18) remains bounded in  $L^1(Q)$  as  $\delta \rightarrow 0$ . We then deduce that  $T_k(u_n)_t - \Delta_p T_k(u_n)$  is a finite measure in  $Q$ . Let us set  $v_n^k = T_k(u_n)_t - \Delta_p T_k(u_n)$ . By (18)–(19) we have

$$\begin{aligned} \int_Q |v_n^k - \mu_n| &\leq \liminf_{\delta \rightarrow 0} \int_Q \left| S_{k,\delta}(u_n)\mu_n + \frac{|\nabla u_n|^p}{\delta} \operatorname{sign}(u_n)\chi_{\{k \leq |u_n| < k+\delta\}} - \mu_n \right| \\ &\leq \int_{\{|u_n|>k\}} |\mu_n| + \limsup_{\delta \rightarrow 0} \int_{\{k \leq |u_n| < k+\delta\}} \frac{|\nabla u_n|^p}{\delta} \leq 2 \int_{\{|u_n|>k\}} |\mu_n|. \end{aligned} \tag{20}$$

Recall that by Proposition 2.2 the sequence  $(\mu_n)$  is  $p$ -equidiffuse. Applying Proposition 1.3 we can fix  $k > 0$  sufficiently large (depending only on  $\varepsilon > 0$ ) so that the right-hand side of (20) is  $\leq \varepsilon$ ,  $\forall n \geq 1$ . In particular,  $(v_n^k)$  remains uniformly bounded in  $\mathcal{M}(Q)$  as  $n \rightarrow \infty$ . Passing to a subsequence, we may assume that  $(v_n^k)$  converges weak\* in  $\mathcal{M}(Q)$  to some measure  $v^k$  as  $n \rightarrow \infty$ . By classical results on parabolic equations with measure data, there exists a function  $u \in L^1(Q)$  such that (taking a further subsequence if necessary)  $u_n \rightarrow u$  and  $\nabla u_n \rightarrow \nabla u$  a.e. on  $Q$  (see [1]). In particular,

$$\begin{aligned} T_k(u_n) &\rightharpoonup T_k(u) \quad \text{weakly in } L^p(0, T; W_0^{1,p}(\Omega)) \quad \text{and} \\ \Delta_p T_k(u_n) &\rightharpoonup \Delta_p T_k(u) \quad \text{weakly in } L^{p'}(0, T; W^{-1,p'}(\Omega)) \end{aligned}$$

as  $n \rightarrow \infty$ . Therefore,  $v^k = T_k(u)_t - \Delta_p T_k(u)$ . Since  $T_k(u) \in L^p(0, T; W_0^{1,p}(\Omega)) \cap L^\infty(Q)$ , it follows from Proposition 1.1 that  $v^k$  is a diffuse measure. Moreover, from (20) and the lower semicontinuity of the norm with respect to the weak\* convergence, we obtain

$$\|v^k - \mu\|_{\mathcal{M}(Q)} \leq \liminf_{n \rightarrow \infty} \int_Q |v_n^k - \mu_n| \leq 2 \liminf_{n \rightarrow \infty} \int_{\{|u_n|>k\}} |\mu_n| \leq \varepsilon. \tag{21}$$

This concludes the proof of Theorem 1.2.  $\square$

We now present an example of diffuse measures for which (6) does not hold with  $g \in L^\infty(Q)$ .

**Example 1.** Assume that  $p = 2$ . Given  $0 < t_0 < T$ , let  $\mu = \delta_{t_0} \otimes f$ , where  $f \in L^1(\Omega)$ . By [3, Theorem 2.15],  $\mu$  is diffuse. We claim that if (6) holds for some  $g \in L^\infty(Q)$ , then  $f \in L^\infty(\Omega)$  and  $\|f\|_{L^\infty(\Omega)} \leq 2\|g\|_{L^\infty(Q)}$ .

Indeed, let  $u \in L^2(t_0, T; H_0^1(\Omega)) \cap C([t_0, T]; L^1(\Omega))$  be the solution of (see [7])

$$\begin{cases} u_t - \Delta u + |\nabla u|^2 = 0 & \text{in } (t_0, T) \times \Omega, \\ u = f & \text{on } \{t_0\} \times \Omega, \\ u = 0 & \text{on } (t_0, T) \times \partial\Omega. \end{cases}$$

Denoting by  $\tilde{u}$  the extension of  $u$  in  $Q$  as identically zero on  $(0, t_0) \times \Omega$ , then  $\tilde{u} \in L^2(0, T; H_0^1(\Omega))$  and

$$\tilde{u}_t - \Delta \tilde{u} + |\nabla \tilde{u}|^2 = \mu \quad \text{in } Q. \tag{22}$$

Then Eq. (22) provides another decomposition of  $\mu$  as in (6). By [3, Lemma 2.29], it follows that  $\tilde{u} - g \in C([0, T]; L^1(\Omega))$ . Since  $u \in C([t_0, T]; L^1(\Omega))$ ,  $u(t_0) = f$ , and  $\tilde{u} = u\chi_{[t_0, T]}$  one then shows that  $\int_\Omega f\varphi \, dx \leq 2\|g\|_{L^\infty(Q)}\|\varphi\|_{L^1(\Omega)} \, \forall \varphi \in C_0^\infty(\Omega)$ , which gives  $f \in L^\infty(\Omega)$  and the claimed estimate.

Thus,  $\mu$  is a diffuse measure which cannot be written as (6) with  $g \in L^\infty(\Omega)$  if  $f \notin L^\infty(\Omega)$ .

### References

[1] L. Boccardo, A. Dall’Aglia, T. Gallouët, L. Orsina, Nonlinear parabolic equations with measure data, *J. Funct. Anal.* 147 (1997) 237–258.  
 [2] H. Brezis, A.C. Ponce, Reduced measures for obstacle problems, *Adv. Differential Equations* 10 (2005) 1201–1234.  
 [3] J. Droniou, A. Porretta, A. Prignet, Parabolic capacity and soft measures for nonlinear equations, *Potential Anal.* 19 (2003) 99–161.  
 [4] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Paris, Dunod, 1969.  
 [5] F. Petitta, A.C. Ponce, A. Porretta, Strong approximation of diffuse measures and nonlinear parabolic equations, in preparation.  
 [6] M. Pierre, Parabolic capacity and Sobolev spaces, *SIAM J. Math. Anal.* 14 (1983) 522–533.  
 [7] A. Porretta, Existence results for nonlinear parabolic equations via strong convergence of truncations, *Ann. Mat. Pura Appl.* (IV) 177 (1999) 143–172.