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C. R. Acad. Sci. Paris, Ser. I 345 (2007) 667–672
COMPTES RENDUS MATHEMATIQUE
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Mathematical Analysis

Dimension of the spectrum of one-dimensional discrete Schrödinger operators with Sturmian potentials

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Received 13 August 2007; accepted 29 October 2007

Available online 26 November 2007

Presented by Jean-Pierre Kahane

Abstract

Damanik and collaborators (2007) gave the behavior for large coupling constant of the box dimension of the spectrum of a one-dimensional discrete Schrödinger operator whose potential is a Sturm sequence associated with the golden ratio. They also show that in this case the Hausdorff and box dimensions coincide (i.e. the spectrum is dimension-regular). This Note aims at giving a simpler proof of the asymptotic property result and to generalize it to the case of any Sturm potential associated with an irrational frequency whose continued fraction expansion has bounded partial quotients. Moreover, we determine the upper box dimension of the spectrum, with large coupling constant, and show that it is not dimension-regular in general. ***To cite this article: Q.-H. Liu et al., C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

Dimension du spectre d'un opérateur de Schrödinger à potentiel sturmien. Le comportement asymptotique pour une grande constante de couplage de la dimension du spectre d'un opérateur de Schrödinger discret dont le potentiel est une suite sturmienne associée au nombre d'or vient d'être obtenu par Damanik et al. (2007). Dans cette Note, nous donnons une démonstration plus simple de ce résultat et l'étendons au cas d'un potentiel sturmien associé à une fréquence irrationnelle dont les quotient partiels de sa décomposition en fraction continue sont bornés. Nous montrons qu'en général les dimensions de Hausdorff et de Minkowski du spectre sont différentes. ***Pour citer cet article : Q.-H. Liu et al., C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Version française abrégée

On se donne un nombre irrationnel β dont le développement en fraction continue est $[0; a_1, a_2, \ldots]$ et l'on considère la suite, indexée par $\mathbb{Z}$

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[1] Supported by National Science Foundation of China, No. 10501002.
[2] Supported by the National Basic Research Program of China, No. 2007CB814800.

1631-073X/$ – see front matter
doi:10.1016/j.crma.2007.10.048

$$V(n)=\lambda\chi_{[1-\beta,1)}(\beta n+\theta \bmod 1),$$

où $\lambda>0$ et $0\leqslant\theta<1$.

On désigne par $\Sigma_{\beta,\lambda}$ le spectre de l'opérateur H agissant sur $\ell^2(\mathbb{Z})$ ainsi défini

$$Hu(n)=u(n+1)+u(n-1)+V(n)u(n).$$

Ce spectre a fait l'objet de nombreuses études [8,1,7,4,3,5,2]. Il a d'abord été montré qu'il est totalement discontinu, puis qu'il est de mesure nulle. Ensuite c'est le problème de l'estimation de sa dimension qui a été l'objet de recherches.

Dans le cas où $\beta=[0;1,1,1,\dots]$ on sait [2] que l'on a

$$\lim_{\lambda\to\infty}(\log\lambda)\,\overline{\dim}_B\,\Sigma_{\beta,\lambda}=\log(1+\sqrt{2}),$$

où $\overline{\dim}_B$ désigne la dimension de Minkowski supérieure.

L'objet de cette Note est de donner une démonstration plus simple de cette formule et de montrer qu'elle est encore valide pour un nombre β à quotients partiels bornés.

Avant d'énoncer le résultat, il est nécessaire d'introduire quelques notations. On supposera dans toute la suite $\lambda>20$. On pose $M=\sup_{n\geqslant1}a_n$, $c=3/(\lambda-8)$, $d=1/(2\lambda(M+2)^3)$, et, pour $0<x\leqslant1$ et $n\geqslant1$,

$$\mathbf{R}_n(x)=\begin{pmatrix}0 & x^{(a_n-1)} & 0\\ (a_n+1)x & 0 & a_nx\\ a_nx & 0 & (a_n-1)x\end{pmatrix}.$$

Soit $f^*(\beta)$ et $f_*(\beta)$ les racines positives des équations

$$\limsup_{n\to\infty}\big\|\mathbf{R}_1(x)\mathbf{R}_2(x)\cdots\mathbf{R}_n(x)\big\|^{1/n}=1\quad\text{et}\quad\liminf_{n\to\infty}\big\|\mathbf{R}_1(x)\mathbf{R}_2(x)\cdots\mathbf{R}_n(x)\big\|^{1/n}=1.$$

On peut alors énoncer les résultats :

Théorème 0.1. *On a*

$$\frac{\log f^*(\beta)}{\log d}\leqslant\overline{\dim}_B\,\Sigma_{\beta,\lambda}\leqslant\frac{\log f^*(\beta)}{\log c}\quad et\quad \dim_H\Sigma_{\beta,\lambda}\leqslant\frac{\log f_*(\beta)}{\log c}.$$

De plus, on a

$$\lim_{\lambda\to\infty}(\log\lambda)\,\overline{\dim}_B\,\Sigma_{\beta,\lambda}=\log f^*(\beta).$$

Théorème 0.2. *On suppose que β est un irrationnel quadratique. Alors, il existe k et $p>0$ tels que $a_n=a_{n+p}$ pour $n>k$. Soit $f(\beta)$ la valeur de x pour laquelle la valeur propre de Perron–Frobenius de la matrice $\mathbf{R}_{k+1}(x)\mathbf{R}_{k+2}(x)\cdots\mathbf{R}_{k+p}(x)$ est 1. Alors, on a $f(\beta)=f_*(\beta)=f^*(\beta)$ et*

$$\lim_{\lambda\to\infty}(\log\lambda)\,\overline{\dim}_B\,\Sigma_{\beta,\lambda}=\log f(\beta).$$

On peut observer que $f(\beta)$ est l'unique racine positive du polynôme $\det(\mathbf{R}_{k+1}(x)\cdots\mathbf{R}_{k+p}(x)-I)$.

Les démonstrations utilisent principalement les ingrédients suivants : la notion de structure naturelle de recouvrement d'une partie compacte de la droite, développée dans [5,6] et la construction explicite de telles structures pour les spectres étudiés [1,7].

1. Introduction

We consider the one-dimensional discrete Schrödinger operator

$$[Hu](n)=u(n+1)+u(n-1)+V(n)u(n)$$

acting on $\ell^2(\mathbb{Z})$ whose potential $V:\mathbb{Z}\to\mathbb{R}$ is given by

$$V(n)=\lambda\chi_{[1-\beta,1)}(n\beta+\theta \bmod 1),$$

where $\lambda > 0$ is the coupling constant, $\theta \in [0, 1)$ the phase, and β, called the frequency, a positive irrational number whose continued fraction expansion is $[0; a_1, a_2, \ldots]$. We denote the spectrum of H by $\Sigma_{\beta,\lambda}$, since it is independent of θ.

In 1987, Sütő [8] showed that for the Fibonacci Hamiltonian, i.e., the case $\beta = [0; 1, 1, \ldots]$, and if $\lambda > 4$, then $\Sigma_{\beta,\lambda}$ is a Cantor set. In 1989, Bellissard et al. [1] proved that for any irrational β and any $\lambda > 0$, $\Sigma_{\beta,\lambda}$ has zero Lebesgue measure. In 1997, Raymond [7] proved that the Hausdorff dimension of the spectrum of the Fibonacci Hamiltonian is smaller than 1. In 2000, Jitomirskaya and Last [4] proved that the spectral measure of the Fibonacci Hamiltonian is α-continuous for some $\alpha > 0$, and so gave a positive lower bound for the Hausdorff dimension of the spectrum. And immediately after, Damanik et al. [3] extended their results to the case when $\beta = [0; a_1, a_2, \ldots]$ with $\underline{\lim}_{n\to\infty} \sum_{i=1}^n a_i/n < \infty$. In 2004, Liu and Wen [5] showed that for $\lambda > 20$, $\dim_H \Sigma_{\beta,\lambda} = 1$ if and only if $\overline{\lim}_{n\to\infty} (\prod_{i=1}^n a_i)^{1/n} < \infty$, and for all the other irrational β, $0 < \dim_H \Sigma_{\beta,\lambda} < 1$.

Recently, in [2], it was shown that the spectrum of Fibonacci Hamiltonian with $\lambda \geqslant 16$ is dimension-regular (i.e., the Hausdorff dimension $\dim_H$ and box dimension $\dim_B$ coincide), and moreover one has

$$\lim_{\lambda\to\infty} (\log \lambda) \dim_B \Sigma_{\beta,\lambda} = \log(1 + \sqrt{2}). \tag{1}$$

Notice that $(1 + \sqrt{2})^{-1}$ is the positive root of $x^2 + 2x - 1 = 0$.

In this note, we give a more direct proof for (1) and extend the result to the case when β has bounded partial quotients. Then we show that the Hausdorff dimension and the upper box dimension ($\overline{\dim}_B$) may be different.

2. Results and proofs

Before studying this spectrum, we recall some facts in [5,6]:

Definition 2.1. Let $E \subset \mathbb{R}$ be a compact set, and $\{\mathcal{F}_k\}_{k\geqslant 0}$ a family of finite coverings of E. Such a family is said to be a natural covering structure (NCS for short) for the set E if it satisfies

(1) for any $k \geqslant 1$, $\mathcal{F}_k$ is composed of closed intervals whose interiors are disjoint;
(2) for any $k \geqslant 1$, any $J \in \mathcal{F}_k$ is contained in a unique interval of $\mathcal{F}_{k-1}$ (we denote it by J^{-1} in this note);
(3) $\lim_{k\to\infty} d_k = 0$, where $d_k = \max_{J\in\mathcal{F}_k} |J|$;
(4) $E = \bigcap_{k\geqslant 1} \bigcup_{J\in\mathcal{F}_k} J$.

Notations. We introduce a few more notations:
$\mathcal{F} = \bigcup_{k\geqslant 1} \mathcal{F}_k$, $c_* = \inf_{J\in\mathcal{F}} \frac{|J|}{|J^{-1}|}$, and $c^* = \sup_{J\in\mathcal{F}} \frac{|J|}{|J^{-1}|}$.
If $\mathcal{U}$ is a finite family of intervals and $s > 0$, we set $\|\mathcal{U}\|_s = \sum_{J\in\mathcal{U}} |J|^s$.

For $k \geqslant 1$, let s_k be the unique real number satisfying $\|\mathcal{F}_k\|_{s_k} = 1$. We set

$$s_* = \liminf_{k\to\infty} s_k, \qquad s^* = \limsup_{k\to\infty} s_k.$$

Sometimes, we shall write these quantities as $s_k(\beta, \lambda)$, $s_*(\beta, \lambda)$ and $s^*(\beta, \lambda)$, once given a NCS for $\Sigma_{\beta,\lambda}$.

Theorem A. *(See [6].) Let $\{\mathcal{F}_k\}_{k\geqslant 0}$ be a NCS for E, then $\dim_H E \leqslant s_*$. Moreover, if $0 < c_* \leqslant c^* < 1$, then $\overline{\dim}_B E = s^*$.*

Remark 1. The condition $c^* < 1$ can be weakened: this theorem still holds true if there exist $m > 0$ and $0 < c < 1$ such that for k large enough and all $J \in \mathcal{F}_k$, one has $|J|/|J^{-m}| < c$. To get Theorem 2.2 we need $m = 2$; also this is the reason why we take the product of two matrices in (5).

Fix $\lambda > 20$ and $\beta = [0; a_1, a_2, \ldots]$. For any integer $n \geqslant 1$, we define three matrices $(t_{ij}(n))$, $(p_{ij}(n))$, and $(q_{ij}(n))$:

$$\begin{pmatrix} 0 & 1 & 0 \\ a_n + 1 & 0 & a_n \\ a_n & 0 & a_n - 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & t_1^{a_n-1} & 0 \\ t_1/a_n & 0 & t_1/a_n \\ t_1/a_n & 0 & t_1/a_n \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & t_2^{a_n-1} & 0 \\ t_2(a_n+2)^{-3} & 0 & t_2(a_n+2)^{-3} \\ t_2(a_n+2)^{-3} & 0 & t_2(a_n+2)^{-3} \end{pmatrix}, \tag{2}$$

where $t_1 = 3/(\lambda - 8)$, $t_2 = 1/(\lambda + 5)$. According to [7] and [5,6], we have a NCS $\{\mathcal{F}_n\}_{n\geqslant 0}$ for $\Sigma_{\beta,\lambda}$, which can be colored with three colors (labeled 1, 2, and 3) so that, for any $n \geqslant 1$,

(i) $\mathcal{F}_0$ consists of the two intervals $[-2, 2]$ and $[\lambda - 2, \lambda + 2]$ whose respective colors are 3 and 1;
(ii) for any $n \geqslant 1$ and $i, j \in \{1, 2, 3\}$, any element of $\mathcal{F}_n$ of color i exactly contains $t_{ij}(n)$ elements of $\mathcal{F}_{n+1}$ of color j;
(iii) for any $n \geqslant 1$ and for any $J \in \mathcal{F}_{n+1}$, we have

$$q_{ij}(n) \leqslant \frac{|J|}{|J^{-1}|} \leqslant p_{ij}(n),$$

where i and j are the respective colors of J^{-1} and J.

A mere consequence of Theorem A is the following:

Theorem 2.2. *Let $\beta = [0; a_1, a_2, a_3, \ldots]$ be irrational with $\{a_n\}_{n\geqslant 1}$ bounded, $\lambda > 20$, and $\{\mathcal{F}_n\}_{n\geqslant 0}$ be the corresponding three-color NCS for $\Sigma_{\beta,\lambda}$, then*

$$\dim_H \Sigma_{\beta,\lambda} \leqslant s_*(\beta, \lambda) \quad \text{and} \quad \overline{\dim}_B \Sigma_{\beta,\lambda} = s^*(\beta, \lambda).$$

Theorem 2.3. *Let $\beta = [0; a_1, a_2, a_3, \ldots]$ be irrational with partial quotients a_n bounded by M. Assume that $\lambda > 20$, and set $c = 3/(\lambda - 8)$, $d = 1/(2\lambda(M + 2)^3)$, and, for any $0 < x \leqslant 1$ and $n \geqslant 1$,*

$$\mathbf{R}_n(x) = \begin{pmatrix} 0 & x^{(a_n-1)} & 0 \\ (a_n + 1)x & 0 & a_n x \\ a_n x & 0 & (a_n - 1)x \end{pmatrix}. \tag{3}$$

Let $f^*(\beta)$ and $f_*(\beta)$ be the positive roots of the equations

$$\limsup_{n\to\infty} \big\| \mathbf{R}_1(x)\mathbf{R}_2(x) \cdots \mathbf{R}_n(x) \big\|^{1/n} = 1 \quad \text{and} \quad \liminf_{n\to\infty} \big\| \mathbf{R}_1(x)\mathbf{R}_2(x) \cdots \mathbf{R}_n(x) \big\|^{1/n} = 1.$$

Then

$$\frac{\log f^*(\beta)}{\log d} \leqslant s^* \leqslant \frac{\log f^*(\beta)}{\log c}, \qquad \frac{\log f_*(\beta)}{\log d} \leqslant s_* \leqslant \frac{\log f_*(\beta)}{\log c}, \tag{4}$$

where s^* and s_* are associated with the above three-color NCS for $\Sigma_{\beta,\lambda}$.

Proof. As in Eq. (5.2) of [5], we have

$$\big\| v\mathbf{R}(d^s) \cdots \mathbf{R}_n(d^s) \big\| \leqslant \|\mathcal{F}_n\|_s \leqslant \big\| v\mathbf{R}_1(c^s) \cdots \mathbf{R}_n(c^s) \big\|,$$

where v is the row vector $(4^s, 0, 4^s)$ (since $\mathcal{F}_0$ is composed of two bands of colors 1 and 3 of length 4). Here, the norm of a vector is the sums of the absolute values of its entries, and the norm of a matrix is the maximum of the norms of its rows.

Fix a small $0 < r < 1$. For any $r < x < 1$, all elements of $\mathbf{R}_1(x) \cdots \mathbf{R}_5(x)$ have common positive lower and upper bounds depending on r and M. So there exists a positive constant η such that, for any $n \geqslant 5$ and $r < x < 1$,

$$\eta \big\| \mathbf{R}_1(x) \cdots \mathbf{R}_n(x) \big\| \leqslant \big\| (1, 0, 1)\mathbf{R}_1(x) \cdots \mathbf{R}_n(x) \big\| \leqslant 2 \big\| \mathbf{R}_1(x) \cdots \mathbf{R}_n(x) \big\|.$$

For any $0 < x < y < 1$ and $n \geqslant 1$, we have $\mathbf{R}_n(x) \leqslant \mathbf{R}_n(y)$ and

$$\mathbf{R}_n(x)\mathbf{R}_{n+1}(x) \leqslant (x/y)\mathbf{R}_n(y)\mathbf{R}_{n+1}(y), \tag{5}$$

which implies that $f^*(\beta)$ and $f_*(\beta)$ are well defined.

If $c^s < f^*(\beta)$ for some $0 < s < 1$, then

$$\|\mathcal{F}_n\|_s \leqslant 2\,4^s \big\| \mathbf{R}_1(c^s) \cdots \mathbf{R}_n(c^s) \big\| \leqslant 2\,4^s \left(\frac{c^s}{f^*(\beta)} \right)^{[n/2]} \big\| \mathbf{R}_1\big(f^*(\beta)\big) \cdots \mathbf{R}_n\big(f^*(\beta)\big) \big\|,$$

so $\|\mathcal{F}_n\|_s < 1$ for n large enough. Hence $s_n \leqslant \log f^*(\beta)/\log c$ and $s^* \leqslant \log f^*(\beta)/\log c$.

If $d^s > f^*(\beta)$ for some $0 < s < 1$, then

$$\|\mathcal{F}_n\|_s \geqslant \eta \big\| \mathbf{R}_1(d^s) \cdots \mathbf{R}_n(d^s) \big\| \geqslant \eta \left(\frac{d^s}{f^*(\beta)} \right)^{[n/2]} \big\| \mathbf{R}_1\big(f^*(\beta)\big) \cdots \mathbf{R}_n\big(f^*(\beta)\big) \big\|,$$

so $\|\mathcal{F}_n\|_s > 1$ infinitely often. Hence $s^* \geqslant \log f^*(\beta)/\log d$.

By a similar argument, we can prove (4) for s_*. □

Remark 2. Eq. (4) implies

$$\lim_{\lambda\to\infty} s_*(\beta,\lambda)\log\lambda = -\log f_*(\beta) \quad \text{and} \quad \lim_{\lambda\to\infty} s^*(\beta,\lambda)\log\lambda = -\log f^*(\beta).$$

Corollary 2.4. *Let β be an irrational whose continued fraction expansion $[0; a_1, a_2, a_3, \ldots]$ is ultimately periodic, i.e., there exist positive k and $p > 0$ such that $a_n = a_{n+p}$ for $n > k$. If $\lambda > 20$ and if $\{\mathcal{F}_n\}_{n\geqslant 0}$ is the corresponding three-color NCS for $\Sigma_{\beta,\lambda}$, then*

$$\frac{\log f(\beta)}{\log d} \leqslant s_* \leqslant s^* \leqslant \frac{\log f(\beta)}{\log c},$$

where $f(\beta) = f_*(\beta) = f^*(\beta)$; moreover, $f(\beta)$ is the unique positive root of the equation

$$\det\big[\mathbf{R}_{k+1}(x)\mathbf{R}_{k+2}(x)\cdots\mathbf{R}_{k+p}(x) - I_3\big] = 0, \tag{6}$$

where I_3 stands for the order 3 identity matrix.

Proof. Let δ be the maximal eigenvalue of the primitive matrix $\mathbf{R}_{k+1}(x)\mathbf{R}_{k+2}(x)\cdots\mathbf{R}_{k+p}(x)$, then

$$\lim_{n\to\infty} \big\| \mathbf{R}_1(x)\mathbf{R}_2(x)\cdots\mathbf{R}_n(x) \big\|^{1/n} = \delta^{1/p}.$$

Set $\delta = 1$, we see $f_*(\beta) = f^*(\beta)$ and $f(\beta)$ is well defined by (6). □

Remark 3. For $\beta = [0; 1, 1, \ldots]$, $f(\beta)$, the positive root of equation

$$\det \begin{bmatrix} -1 & 1 & 0 \\ 2x & -1 & x \\ x & 0 & -1 \end{bmatrix} = -1 + 2x + x^2 = 0,$$

is $\sqrt{2} - 1$. So $-\log f(\beta) = \log(1 + \sqrt{2})$ which proves (1).

Remark 4. Let $\beta_1 = [0; 1, 1, \ldots]$, $\beta_2 = [0; 3, 3, \ldots]$, so $f(\beta_1) > f(\beta_2)$ by direct computation. There exist an increasing sequence of integers $\{m_k\}_{k\geqslant 1}$ with $m_1 = 1$, such that if let $\beta = [0; a_1, a_2, \ldots]$ with $a_n = 1$ for $m_{2k-1} \leqslant n < m_{2k}$, $a_n = 3$ for $m_{2k} \leqslant n < m_{2k+1}$, then

$$f_*(\beta) = f(\beta_2) < f(\beta_1) = f^*(\beta).$$

So for sufficiently large λ, by Theorem 2.2, and Remark 2, we have

$$\dim_H \Sigma_{\beta,\lambda} < \overline{\dim}_B \Sigma_{\beta,\lambda}.$$

Acknowledgements

Liu Qinghui gratefully acknowledges the hospitality of the Mathematics Department of the Université Paris-Sud where part of this work was done.

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