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Probability Theory

Large deviation principle for a backward stochastic differential equation with subdifferential operator

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Abstract

In this note, we prove that the solution of a backward stochastic differential equation, which involves a subdifferential operator and associated to a family of reflecting diffusion processes, converges to the solution of a deterministic backward equation and satisfies a large deviation principle. ***To cite this article: E.H. Essaky, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Un principe de grandes déviations pour une équation différentielle stochastique rétrograde associée à un opérateur sous-différentiel. Dans cette Note, nous montrons que la solution d'une équation différentielle stochastique rétrograde progressive associée à un opérateur sous-différentiel converge vers la solution d'une équation différentielle rétrograde progressive déterministe et satisfait un principe de grandes déviations. ***Pour citer cet article : E.H. Essaky, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

1. Introduction, notations and assumptions

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\leqslant 1}))$ be a stochastic basis such that $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$, $\mathcal{F}_{t+} = \mathcal{F}_t$, $\forall t \leqslant 1$, and suppose that the filtration is generated by a d-dimensional Brownian motion B. On other hand, let:

• Θ be an open connected bounded subset of $\mathbb{R}^d$, which is such that for a function $\psi \in \mathcal{C}_b^2(\mathbb{R}^d)$, $\Theta = \{\psi > 0\}$, $\partial\Theta = \{\psi = 0\}$, and $|\nabla\psi(x)| = 1$, $x \in \partial\Theta$. Note that at any boundary point $x \in \partial\Theta$, $\nabla\psi(x)$ is a unit normal vector to the boundary, pointing towards the interior of Θ. The above assumptions imply that there exists a constant $\delta > 0$ such that for all $x \in \partial\Theta$, $x' \in \overline{\Theta}$

$$2\langle x' - x, \nabla\psi(x)\rangle + \delta|x - x'|^2 \geqslant 0. \tag{1}$$

• $b : \overline{\Theta} \to \mathbb{R}^d$, $\sigma : \overline{\Theta} \to \mathbb{R}^{d\times d}$ be functions such that:

(A1) There exists a constant $C > 0$ such that

$$\left|b(x)\right| + \left|\sigma(x)\right| \leqslant C, \quad \left|b(x) - b(x')\right| + \left|\sigma(x) - \sigma(x')\right| \leqslant C|x - x'|, \quad \forall x, x' \in \overline{\Theta}.$$

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(A2) There exists a constant $\gamma > 0$ such that: $a(x) \geqslant \gamma|x|^2, \forall x \in \overline{\Theta}$.
• $h \in \mathcal{C}(\overline{\Theta}; \mathbb{R}^k)$, $f \in \mathcal{C}([0,1] \times \overline{\Theta} \times \mathbb{R}^k \times \mathbb{R}^{k\times d}; \mathbb{R}^k)$ be functions satisfying the following assumptions:
(A3) There exist constants $\alpha \in \mathbb{R}$, $K > 0$, $c > 0$ and $\mu > 0$ such that

$$\text{(i)}\quad \forall t, \forall x, \forall y, \forall (z, z'), \quad \left|f(t,x,y,z) - f(t,x,y,z')\right| \leqslant \mu|z - z'|,$$

$$\text{(ii)}\quad \forall t, \forall x, \forall z, \forall (y, y'), \quad \left\langle y - y', f(t,x,y,z) - f(t,x,y',z)\right\rangle \leqslant \alpha|y - y'|^2,$$

$$\text{(iii)}\quad \forall x, \forall x', \quad \left|h(x) - h(x')\right| \leqslant c|x - x'|,$$

$$\text{(vi)}\quad \forall t, \forall x, \forall y, \forall z, \quad \left|f(t,x,y,z)\right| \leqslant K\big(1 + |y| + |z|\big), \quad (v)\forall x, \quad \left|h(x)\right| \leqslant K\big(1 + |x|\big).$$

• $\Pi : \mathbb{R}^k \to]-\infty, +\infty]$, be a proper lower semicontinuous convex function such that
(A4) There exists a constant $C > 0$ such that: $\Pi(h(x)) \leqslant C(1 + |x|),\ \forall x \in \overline{\Theta},\ \Pi(y) \geqslant \Pi(0) = 0,\ \forall y \in \mathbb{R}^k$.

We need also the following notations:
• $\mathcal{C}[0,T]$ denotes the space of continuous functions $\Phi : [0,T] \to \mathbb{R}^d$ such that $f(0) \in \overline{\Theta}$.
• $\overline{\mathcal{C}}[0,T]$ denotes the space of continuous functions $\Psi : [0,T] \to \overline{\Theta}$.
• $\mathcal{V}[0,T]$ denotes the space of functions $\rho : [0,T] \to \mathbb{R}^d$ with bounded variation and $\rho(0) = 0$.
• $\delta\Pi$ denotes the subdifferential operator of the function Π and is defined by

$$\delta\Pi(u) = \left\{u^* \in \mathbb{R}^k\colon\ \langle u^*, v - u\rangle + \Pi(u) \leqslant \Pi(v), \forall v \in \mathbb{R}^k\right\}.$$

Note that the subdifferential operator $\delta\Pi : \mathbb{R}^k \to 2^{\mathbb{R}^k}$ is a maximal monotone operator, that is

$$\langle u' - v', u - v\rangle \geqslant 0 \quad \forall (u, u'), (v, v') \in \delta\Pi. \tag{2}$$

For $\rho \in \mathcal{V}[0,T]$, $|\rho|_t$ denotes the total variation of ρ in the interval $[0,t]$.

Consider the system of decoupled forward-backward stochastic differential equations

$$\begin{cases} X_t^{s,x,\varepsilon} = x + \int_s^t b(X_r^{s,x,\varepsilon})\,dr + \sqrt{\varepsilon}\int_s^t \sigma(X_r^{s,x,\varepsilon})\,dB_r + \rho_t^{s,x,\varepsilon} - \rho_s^{s,x,\varepsilon}, \quad 0 \leqslant s \leqslant t \leqslant T, \\ \rho_t^{s,x,\varepsilon} = \int_0^t \nabla\psi(X_r^{s,x,\varepsilon})\,d|\rho^{s,x,\varepsilon}|_r, \quad |\rho^{s,x,\varepsilon}|_t = \int_0^t \mathbf{1}_{\{X_r^{s,x,\varepsilon} \in \partial\Theta\}}\,d|\rho^{s,x,\varepsilon}|_r, \end{cases} \tag{3}$$

$$\begin{cases} Y_t^{s,x,\varepsilon} = h(X_T^{s,x,\varepsilon}) + \int_t^T f(r, X_r^{s,x,\varepsilon}, Y_r^{s,x,\varepsilon}, Z_r^{s,x,\varepsilon})\,dr - \int_t^T Z_r^{s,x,\varepsilon}\,dB_r - \int_t^T U_r^{s,x,\varepsilon}\,dr, \\ (Y_t^{s,x,\varepsilon}, U_t^{s,x,\varepsilon}) \in \delta\Pi, \quad \text{and} \quad \mathbb{E}\int_0^T \Pi(Y_r^{s,x,\varepsilon})\,dr < +\infty. \end{cases} \tag{4}$$

The existence and uniqueness of the strong solution $X^{s,x,\varepsilon}$, under assumption **(A1)**, for Eq. (3) is standard (see, for example, Lions and Sznitman [2] or Saisho [5]). It follows also from the result of Pardoux and Rascanu [3] that, under assumptions **(A3)** and **(A4)**, there exists a unique triple $(Y^{s,x,\varepsilon}, Z^{s,x,\varepsilon}, U^{s,x,\varepsilon})$ for Eq. (4).

The objective of this work is to prove that the solution of decoupled forward-backward stochastic differential equations (3)–(4) converges, as ε goes to 0, to the solution $(\chi^{s,x}, \rho^{s,x}, Y^{s,x}, Z^{s,x}, U^{s,x})$ of the following decoupled forward-backward deterministic equation:

$$\begin{cases} \chi_t^{s,x} = x + \int_s^t b(\chi_r^{s,x})\,dr + \rho_t^{s,x} - \rho_s^{s,x}, \quad \rho_t^{s,x} = \int_0^t \nabla\psi(\chi_r^{s,x})\,d|\rho^{s,x}|_r, \quad |\rho^{s,x}|_t = \int_0^t \mathbf{1}_{\{\chi_r^{s,x} \in \partial\Theta\}}\,d|\rho^{s,x}|_r, \\ Y_t^{s,x} = h(\chi_T^{s,x}) + \int_t^T f(r, \chi_r^{s,x}, Y_r^{s,x}, 0)\,dr - \int_t^T U_r^{s,x}\,dr, \quad (Y_t^{s,x}, U_t^{s,x}) \in \delta\Pi, \\ \quad \text{and} \quad \mathbb{E}\int_0^T \Pi(Y_r^{s,x})\,dr < +\infty, \end{cases} \tag{5}$$

and satisfies a large deviation principle. Our result is, in fact, a generalization of the work by Rainero [4] where the case of $(\rho^{s,x,\varepsilon}, U^{s,x,\varepsilon}, \Pi) = (0,0,0)$ has been considered.

For the sake of simplicity, we put, in general, $s = 0$. Of course, the results hold true for all $s \in [0,T]$. We denote then by $X^{x,\varepsilon} := X^{0,x,\varepsilon}$, $Y^{0,x,\varepsilon} := Y^{x,\varepsilon}, \ldots$.

2. Large deviation principle and convergence of the solution of the forward equation

Let $\Phi \in \mathcal{C}[0,T]$, $\Psi \in \overline{\mathcal{C}}[0,T]$, $\rho \in \mathcal{V}[0,T]$ such that

$$\Psi(t) = \Phi(t) + \rho(t), \quad \rho_t = \int_0^t \nabla\psi(\Psi_r)\,d|\rho|_r,\ |\rho|_t = \int_0^t \mathbf{1}_{\{\Psi(r) \in \partial\Theta\}}\,d|\rho|_r. \tag{6}$$

For Φ and Ψ defined as above, let $\Psi = F(\Phi)$. It is known from Lions and Sznitman [2] or Saisho [5] that F is continuous. We have the following theorem:

Theorem 2.1. *The process $X^{x,\varepsilon}$ given by Eq.* (3) *satisfies, as ε goes to* 0*, a large deviation principle with rate function $S(\Psi)$ defined by*: $S(\Psi) = \frac{1}{2}\inf_{\Phi\in F^{-1}(\Psi)}\int_0^T(\dot\Phi(s) - b(\Psi(s)))^* a^{-1}(\Psi(s))(\dot\Phi(s) - b(\Psi(s)))\,ds$, *with the fact that $S(\Psi) = \infty$ if $F^{-1}(\Psi) = \emptyset$ or Φ is not absolutely continuous.*

Proof. The result follows by using the contraction principle (see Dembo and Zeitouni [1]) and a large deviation principle for diffusion processes (see Stroock [7] or [1], see Sheu [6] for other assumptions on Θ). □

Applying Itô's formula to $e^{-\delta(\psi(X_t^{x,\varepsilon})+\psi(\chi_t^x))}|X_t^{x,\varepsilon} - \chi_t^x|^2$, where δ is given by the inequality (1), and using the boundedness of b, σ, ψ, $\nabla\psi$, $D^2\psi$, and Burkholder–Davis–Gundy inequality, we have:

Lemma 2.2. *For all $\varepsilon \in\,]0,1]$, there exists a constant $C > 0$, independent of x and ε, such that*

$$\mathbb{E}\sup_{0\leqslant t\leqslant T}|X_t^{x,\varepsilon} - \chi_t^x|^2 \leqslant C\varepsilon.$$

Remark 1. As a consequence of Lemma 2.2, the solution of the reflecting diffusion process $X^{x,\varepsilon}$ converges to the deterministic path χ^x in L^2.

3. Convergence and large deviation principle for the solution of the backward equation

Let $(\chi^{(s,x)}, \rho^{s,x}, Y^{(s,x)}, 0, U^{(s,x)})$ be the solution of deterministic equation (5).

Applying Itô's formula to $|Y_t^{s,x,\varepsilon} - Y_t^{s,x}|^2$, and using inequality (2), assumption **(A3)** and Burkholder–Davis–Gundy inequality we get the following:

Lemma 3.1. $\forall\varepsilon \in\,]0,1]$, *there exists a constant $C > 0$, independent of s, x and ε, such that*

$$\mathbb{E}\left[\sup_{s\leqslant t\leqslant T}|Y_t^{s,x,\varepsilon} - Y_t^{s,x}|^2 + \int_s^T |Z_r^{s,x,\varepsilon}|^2\,dr\right] \leqslant C\left[\mathbb{E}\big(X_T^{s,x,\varepsilon} - \chi_T^{s,x}|^2\big) + \mathbb{E}\int_s^T |X_r^{s,x,\varepsilon} - \chi_r^{s,x}|^2\,dr\right].$$

Remark 2. As a consequence of Lemmas 3.1 and 2.2, we get

$$\mathbb{E}\left[\sup_{s\leqslant t\leqslant T}|Y_t^{s,x,\varepsilon} - Y_t^{s,x}|^2 + \int_s^T |Z_r^{s,x,\varepsilon}|^2\,dr\right] \leqslant C\varepsilon,$$

where C is a positive constant and then the solution of the BSDE (4) converges to the deterministic solution of the backward equation of system (5).

We now consider the BSDE in the case $k = 1$. We want to prove that the process $Y^{s,x,\varepsilon}$ satisfies a large deviation principle. For that reason, we recall the link between Variational Inequality (VI, for short) and BSDE. For all $\varepsilon \geqslant 0$, we consider the following VI

$$\begin{cases} \frac{\partial u^\varepsilon}{\partial t}(t,x) + \mathcal{L}^{x,\varepsilon}u^\varepsilon(t,x) + f(t,x,u^\varepsilon(t,x),((\nabla u^\varepsilon)^*\sqrt{\varepsilon}\sigma)(t,x)) \in \delta\Pi(u^\varepsilon(t,x)), \quad t\in\,]0,T[,\ x\in\Theta,\\ \frac{\partial u^\varepsilon}{\partial n}(t,x) \in \delta\Pi\big(u^\varepsilon(t,x)\big), \quad x\in\partial\Theta, \quad u^\varepsilon(T,x) = h(x), \quad x\in\overline{\Theta}, \end{cases} \tag{7}$$

where $\mathcal{L}^{x,\varepsilon} := \frac{\varepsilon}{2}\sum_{i,j=1}^d(\sigma\sigma^*)_{ij}\frac{\partial^2}{\partial x_i\partial x_j} + \sum_{i=1}^d b_i\frac{\partial}{\partial x_i}$ is the second order partial differential operator, and at point $x\in\partial\Theta$, $\frac{\partial}{\partial n} := \sum_{i=1}^d\frac{\partial\psi}{\partial x_i}(x)\frac{\partial}{\partial x_i}$. Then we have, for each $(t,x)\in[0,T]\times\overline{\Theta}$, $u^\varepsilon(t,x) = Y_t^{t,x,\varepsilon}$, both in the sense that any classical solution of the VI (7) is equal to $Y_t^{s,x,\varepsilon}$, and $Y_t^{s,x,\varepsilon}$ is, in the case where all coefficients are continuous, a viscosity solution of the VI (7) (see Pardoux and Rascanu [3]). Moreover, we have also that $Y_t^{s,x,\varepsilon} = u^\varepsilon(t, X_t^{s,x,\varepsilon})$. Let $s \in [0,T]$ and $\varepsilon \geqslant 0$, we define the following applications:

$$F^\varepsilon(\Psi) := \big[t \to u^\varepsilon(t, \Psi_t)\big], \quad t \in [s,T],\ \Psi \in \bar{\mathcal{C}}\big([s,T]\big) \text{ satisfying Eq. (6).}$$

Hence $Y_t^{s,x,\varepsilon} = F^\varepsilon(X^{s,x,\varepsilon})(t)$, for all $t \in [0,T]$, and $Y^{s,x,\varepsilon} = F^\varepsilon(X^{s,x,\varepsilon})$. For $\varepsilon = 0$, u and F stand for u^0 and F^0. We have the following theorem:

Theorem 3.2. *$Y^{x,\varepsilon}$ satisfies, as ε goes to 0, a large deviation principle with a rate function*

$$S'(\Psi') = \inf\big\{S(\Psi) \mid \Psi'_t = F(\Psi)(t) = u(t, \Psi_t), \forall t \in [0,T]\big\}.$$

Proof. In order to apply the same method as for the proof of the contraction principle in Varadhan [8], we just need to show that F^ε, $\varepsilon \geqslant 0$ are continuous and $\{F^\varepsilon\}$ converges uniformly to F on every compact of $\bar{\mathcal{C}}[0,T]$, as ε goes to 0 (see [4]). In fact, since u^ε is continuous, it is not hard to prove that F^ε is also continuous. The uniform convergence of $\{F^\varepsilon\}$ is a consequence of Remark 2. Indeed, let K be a compact of $\bar{\mathcal{C}}[0,T]$, and $G = \{\Psi_s, \Psi \in K, s \in [0,T]\}$. Note that G is a compact of $\overline{\Theta}$. Hence, from Remark 2, we get

$$\sup_{\Psi \in K} \big\| F^\varepsilon(\Psi) - F(\Psi) \big\| = \sup_{\Psi \in K} \sup_{s \in [0,T]} \big|Y_s^{s,\Psi_s,\varepsilon} - Y_s^{s,\Psi_s}\big| \leqslant \sup_{x \in G} \sup_{s \in [0,T]} \big|Y_s^{s,x,\varepsilon} - Y_s^{s,x}\big| \leqslant C\varepsilon.$$

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