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Logic

Claws in digraphs

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Abstract

We study in this Note the existence of claws in digraphs. We extend a result of Saks and Sós to the tournament-like digraphs. ***To cite this article: A. El Sahili, M. Kouider, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Arbres simples dans les graphes orientés. Nous étudions dans cette Note le problème de l'existence des arbres simples dans les graphes orientés. Nous étendons un resultat de Saks et Sós aux graphes orientés presque complets. ***Pour citer cet article : A. El Sahili, M. Kouider, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

1. Introduction

The digraphs considered here are all connected, they have no loops, multiple edges or circuits of length two. The chromatic number of a digraph is the chromatic number of its underlying graph. A *block* of a path in a digraph is a maximal directed subpath. We recall that the *length* of a path is the number of its edges.

A *rooted tree* is a tree T with a specified vertex, called the *root* of T. A *branching* (resp. *inbranching*) is an orientation of a rooted tree in which every vertex except the root has in degree 1 (resp. out degree 1). The *level* of a vertex in a branching is its distance from the root. A *claw* is an inbranching in which only the root may have an in degree more than 2. It will be denoted by $P(s_1, s_2, \ldots, s_k)$ if it is formed by k directed paths ending at the root, of lengths $s_1, s_2, \ldots, s_k$ respectively. In particular, $P(k, l)$ is a path with two blocks of lengths k and l.

Saks and Sós proved the following result about claws in tournaments:

Theorem 1. *(See [5].) Any $2(n-1)$-tournament contains any claw of order n.*

This result became a corollary of that one proved later by Havet and Thomassé [3] showing that a $2(n-1)$-tournament contains any branching of order n.

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By forbidding some paths, we proved [1] the following:

Theorem 2. *Let D be an $2(n-1)$-chromatic digraph without paths $P(2n-3,1)$. Then D contains any claw of order n.*

This gives a direct proof of Saks and Sós' theorem by remarking that the path $P(2n-3,1)$ contains $2n-1$ vertices.

In order to generalize the Saks and Sós' result to the arbitrary digraphs, we want in this paper to extend it to some type of digraphs, the tournament-like digraphs (defined below). We treat in particular the case when the claw is a path with two blocks.

2. Spanning branching forest

A *branching forest* is a digraph each of whose components is a branching. The level of a vertex v in a branching forest F, denoted by $l_F(v)$, is the order of the directed path in F ending at v from the root of the branching of F containing v. The set $L_i(F)$ denotes the set of all vertices of level i in D. We denote by $l(F)$ the maximal integer i such that $L_i(F)$ is non-empty. Again if v is a vertex in F, we denote by $T_v(F)$ the branching of F having v as root.

Definition 1. A spanning branching forest F of a digraph D is said to be maximal if the sum $\sum_{v\in D} l_F(v)$ is maximal.

Definition 2. A tournament-like digraph D is a digraph containing a maximal spanning branching forest F such that $l(F)=\chi(D)$.

An example of a tournament-like digraph which is not a tournament can be obtained easily from any n-chromatic simple graph G: Put $V(G)=\bigcup_{i=1}^{n} S_i$ where S_i are stable sets in G. An edge $x_ix_j \in E(G)$, $x_i \in S_i$, $x_j \in S_j$ is oriented from x_i to x_j whenever $i<j$. The obtained acyclic digraph has no directed path of length n so it is a tournament-like digraph. An example with circuit (see [2]) can be formed by an odd oriented cycle C containing no directed paths of length 3, such that any vertex in C is joined to all the vertices of a strongly connected tournament on $n-3$ vertices.

Lemma 1. *Let F be a maximal spanning branching forest of a digraph D. Then the sets $L_i(F)$, $i \geqslant 0$, are stable sets in D.*

By simply remarking that any vertex in $L_i(F)$ is the terminus of a unique directed path of length $i-1$ in F, the Roy–Gallai theorem [2–4] will be a simple consequence of Lemma 1.

Lemma 1 is also used in [1] to prove the following theorem:

Theorem 3. *Every $n+1$-chromatic digraph contains any path with two blocs of length $n-1$.*

The following property of a maximal spanning branching forest of a digraph D will be useful in the sequel:

Lemma 2. *Let F be a maximal spanning branching forest of a digraph D. Let $e=(x,y)\in E(D)$ with $l_F(x)>l_F(y)$, then F contains a directed yx-path.*

Consequently, we have the following property:

Lemma 3. *Let F be a maximal spanning branching forest of a digraph D. Set $l(F)=l$. For every $y\in L_l(F)$ and for every $r<l$, we have either $N(y)\cap L_r(F)=N^+(y)\cap L_r(F)$ or $N(y)\cap L_r(F)=N^-(y)\cap L_r(F)$.*

We define now an important operation on maximal forests:

Lemma 4. *Let F be a maximal spanning branching forest of a digraph D with $l(F)=l$. Let $C=v_rv_{r+1}\cdots v_{l-1}v_l$ be a circuit of D such that $v_i\in L_i(F)$, $1<r\leqslant i\leqslant l$. Suppose that there is a vertex $x\in L_{r-1}(F)$ such that $e=(x,v_l)\in E(D)$. Then the forest F' obtained from F by rotating the cycle C in the negative sense, that is $F'=$*

$F + e + (v_l, v_r) - (v_{l-1}, v_l) - f$, where f denotes the arc of F with head v_r, is a maximal spanning branching forest and so $T_{v_r}(F) = v_r v_{r+1} \cdots v_{l-1} v_l$ and $l(F') = l$.

F' is called a rotating of F, it is denoted by $\Im(F)$ or $\Im_{C,x}(F)$ for more precision.

Note that the above operation can be also defined even if $r = 1$ by simply rotating C, that is $F' = F + (v_l, v_r) - (v_{l-1}, v_l)$.

Consider a maximal spanning branching forest F of a digraph D with $l(F) = l$. A vertex $y \in L_l(F)$ is said to have a root r, we write $r(y) = r$ if $N^+(y) \cap L_r(F) \neq \phi$ and $N^-(y) \cap L_i(F) \neq \phi$ for all $i \leqslant r - 1$, if any. Set $U_r(F) = \{y \in L_l(F), r(y) = r\}$, $R(D) = \bigcup_{F,r} U_r(F)$ and $\Re(D) = \{r(y), y \in R(D)\}$. All these sets may be empty.

Lemma 5. *Let F be a maximal spanning branching forest of a digraph D with $l(F) = l$. Let $u \in U_r(F)$ for some $r < l$, and let $C = v_r v_{r+1} \cdots v_{l-1} u$ be the circuit of D such that $v_i \in L_i(F)$, $r \leqslant i \leqslant l - 1$. If $F' = \Im(F)$ for some rotating of F using C, then we have: $U_t(F) = U_t(F')$ for all $t < r$ and either $U_r(F') = U_r(F) - \{u\}$ or $U_r(F') = (U_r(F) - \{u\}) \cup \{v_{l-1}\}$.*

If $\Re(D) \neq \phi$, set $\Re(D) = \{r_0, r_1, \ldots, r_s\}$ with $r_0 < r_1 < \cdots < r_s$, $\mathcal{F} = \{F$ maximal, $l(F)$ minimal$\}$ and define the following decreasing sequence of non-empty classes of maximal forests: $\mathcal{F}_0 = \{F \in \mathcal{F}, |U_{r_0}(F)|$ is minimal$\}$ and $\mathcal{F}_{i+1} = \{F \in \mathcal{F}_i, |U_{r_{i+1}}(F)|$ is minimal$\}$, $1 \leqslant i < s$. Based on the above lemma we may remark easily that $\mathcal{F}_k$, $1 \leqslant k \leqslant s$ is closed by some one of the rotating operations:

Lemma 6. *Let $F \in \mathcal{F}_k$, $1 \leqslant k \leqslant s$. Let u be a vertex in $U_{r_k}(F)$, let $C = v_{r_k} v_{r_k+1} \cdots v_{l-1} u$ be the circuit of D such that $v_i \in L_i(F)$, $r_k \leqslant i \leqslant l - 1$. If $F' = \Im(F)$ for some rotating of F using C, then $F' \in \mathcal{F}_k$ and $U_{r_k}(F') = (U_{r_k}(F) - \{u\}) \cup \{v_{l-1}\}$.*

Forests in $\mathcal{F}_k$ have the following crucial property:

Lemma 7. *Let $F \in \mathcal{F}_s$. Let u be a vertex in $U_{r_k}(F)$, $1 \leqslant k \leqslant s$, let $C = v_{r_k} v_{r_k+1} \cdots v_{l-1} u$ be the circuit of D such that $v_i \in L_i(F)$, $r_k \leqslant i \leqslant l - 1$. Then for all $v \in C$ we have $N^-(v) \cap L_j(F) \neq \phi$ for $j < r_k$, and $N(v) \cap (L_j(F) - C) = \phi$ for $j \geqslant r_k$.*

3. Claws

Consider a tournament-like digraph D and let F be a maximal spanning branching forest F such that $l(F) = \chi(D)$. Tournament-like digraphs have the following interesting property:

Lemma 8. *A tournament-like digraph D has a maximal forest F and a tournament T with $V(T) = \{v_r, v_{r+1}, \ldots, v_{l-1}, v_l\}$ such that $v_i \in L_i(F)$, $N^-(v_i) \cap L_j(F) \neq \phi$ $r \leqslant i \leqslant l$ and $j < r$.*

Proof. Let $F \in \mathcal{F}_s$. We have $l = l(F) = \chi(D)$. We suppose that $L_l(F)$ contains no vertex v such that $N^-(v) \cap L_j(F) \neq \phi$ for all $j < l$ since otherwise we take $V(T) = \{v\}$. There is a $k \in \{1, 2, \ldots, s\}$ such that $U_{r_k}(F) \neq \phi$ since otherwise $L_l(F) \cap R(D) = \phi$ and so, due to the first supposition, for all $u \in L_l(F)$ there exists $j < l$ such that $N(u) \cap L_j(F) = \phi$. Hence the vertices in $L_l(F)$ may be distributed over the other levels and as $l(F) = \chi(D)$, we obtain a $(\chi(D) - 1)$-coloring. A contradiction. Suppose to the contrary that the lemma does not hold. Consider a vertex $u \in U_{r_k}(F)$ and let $C = v_{r_k} v_{r_k+1} \cdots v_{l-1} v_l = u$ be the circuit of D such that $v_i \in L_i(F)$, $r_k \leqslant i \leqslant l$. Since $D(V(C))$ is not a tournament then there exist i and j, $r_k \leqslant i$, $j \leqslant l$, such that $v_i v_j \notin E(G(D))$. By defining the periodic sequence of maximal forests in $\mathcal{F}_s$ as in Lemma 7, we may replace F by F_{l-i} which is always in $\mathcal{F}_s$. The vertex u is replaced by v_i. Now $r = l_{F_{l-i}}(v_j) \geqslant r_k$. By Lemma 7, $N(v_i) \cap (L_r(F_{l-i}) - \{v_j\} = \phi$. Then $N(v_i) \cap L_r(F_{l-i}) = \phi$ as $v_i v_j \notin E(G(D))$. If we proceed similarly for all the vertices in $\bigcup_{i=1}^{s} U_{r_i}(F)$, we get a maximal forest $F' \in \mathcal{F}_s$ such that for all $u \in L_l(F')$ there exists $j < l$ such that $N(u) \cap L_j(F') = \phi$ which leads to a contradiction. This completes the proof of the lemma. □

We prove now our main result:

Theorem 4. *A tournament-like digraph D with $\chi(D) = 2n - 2$ contains any claw on n vertices.*

Proof. Let $C = P(s_1, s_2, \ldots, s_k)$ be a claw of order n $(s_0 + s_1 + \cdots + s_k = n - 1)$ with $s_0 = 0$. Let T be a tournament on t vertices and F be a maximal forest as in the above lemma. Let i, $0 \leqslant i \leqslant k$, be the maximal integer such that $s_0 + s_1 + \cdots + s_i \leqslant \frac{t}{2}$. The case $i = k$ corresponds to the case $t = 2n - 2$ and so by the above lemma D is a tournament on $2n - 2$ vertices, it contains a copy of C by Theorem 1. Then we suppose that $i < k$, we have $s_0 + s_1 + \cdots + s_i + s = \frac{t}{2}$ with $0 \leqslant s < s_{i+1}$. Let s' and r be such that $s + s' = s_{i+1}$ and $r + t = 2n - 2$. We have $s' + s_{i+2} + \cdots + s_k = \frac{r}{2} < r$. Thus T is a tournament on $t = 2(s_0 + s_2 + \cdots + s_i + s)$ vertices, it contains a copy of the claw $C' = P(s_0, s_1, \ldots, s_i, s)$. Call v_0, the root of C' and let v be the tail of the leaf corresponding to the path of C' of length s. By the above lemma, there is an in neighbor u of v in $L_{r-1}(F)$. Consider a directed w_1u-path P_1 of length $s' - 1$ in F where $w_1 \in L_{r-s'}(F)$. The path P_1 together with $w_1uv \cdots v_0$ form the $(i+1)$th branch of C. Suppose now that the $(i+j)$th branch of C is constructed without using vertices in $L_h(F)$, $h < a = r - (s' + s_{i+2} + \cdots + s_{i+j})$. if $i + j < k$ then $a \geqslant s_{i+j+1}$ and so if we consider an in neighbor of v_0 in $L_a(F)$, we may construct the $(i+j+1)$th branch of C without using vertices in $L_h(F)$, $h < a - s_{i+j+1} = r - (s' + s_{i+2} + \cdots + s_{i+j+1})$. This induction completes the construction of a copy of C in D. □

A natural problem is to ask about the characterization of tournament-like digraphs.

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