Numerical Analysis

An accurate $H(\text{div})$ flux reconstruction for discontinuous Galerkin approximations of elliptic problems

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Abstract

We introduce a new $H(\text{div})$ flux reconstruction for discontinuous Galerkin approximations of elliptic problems. The reconstructed flux is computed elementwise and its divergence equals the $L^2$-orthogonal projection of the source term onto the discrete space. Moreover, the energy-norm of the error in the flux is bounded by the discrete energy-norm of the error in the primal variable, independently of diffusion heterogeneities.

Résumé


1. Introduction

The approximation of elliptic problems by the discontinuous Galerkin (dG) method has been introduced in the late 1970s and has been, more recently, the subject of extensive research; see, e.g., [2,6] and references therein. Advantages of dG methods include flexibility in the design of approximation spaces (allowing for nonmatching meshes and variable polynomial degree), compact discretization stencils amenable to parallelization, and, in the spirit of finite volumes, a local (elementwise) formulation in terms of numerical fluxes. An issue that still deserves further investigation is whether an accurate $H(\text{div})$ flux reconstruction can be performed using the discrete solution provided by the dG method. This type of postprocessing is important at least in two instances. Firstly, this flux can serve as input data in
The reconstructed flux is proven to be accurate in the space and to use the mean values of the gradient of the dG solution at interfaces to specify the degrees of freedom. The idea therein is to reconstruct the flux in the Brezzi–Douglas–Marini finite element space with the bilinear form

\[ \delta (K \nabla u) = f \quad \text{in } \Omega, \]

\[ u = 0 \quad \text{on } \partial \Omega, \]

with (for simplicity) homogeneous Dirichlet boundary conditions. Here, \( \Omega \subset \mathbb{R}^d \), \( d = 2, 3 \), is a polygonal domain, \( K \in [L^\infty(\Omega)]^{d,d} \) is the diffusion tensor, and \( f \in L^2(\Omega) \) is the source term. The diffusion tensor is assumed to be symmetric and uniformly positive definite in \( \Omega \).

Let \( \{T_h\}_{h>0} \) be a conforming, shape-regular family of affine meshes of \( \Omega \) consisting of simplices. The diffusion tensor is assumed to be piecewise constant on \( T_h \). On an element \( T \in T_h \), the maximal and minimal eigenvalues of \( K \) are denoted by \( \lambda_K(T) \) and \( \Lambda_K(T) \), respectively. For any integer \( k \geq 0 \), consider the usual dG approximation space \( V^h_k = \{ v_h \in L^2(\Omega); \forall T \in T_h, v_h|_T \in \mathbb{P}_k(T) \} \), where \( \mathbb{P}_k(T) \) is the set of polynomials of total degree less than or equal to \( k \).

The L2-scalar product and its associated norm on a subset \( R \subset \Omega \) are indicated by the subscript 0, R. The L2-orthogonal projection from \( L^2(\Omega) \) onto \( V^h_k \) is denoted by \( \Pi^h_k \). Interior and boundary faces are collected in the sets \( F^i_h \) and \( F^b_h \), respectively, and we set \( F_h = F^i_h \cup F^b_h \). For \( F \in F^i_h \), there are \( T^- \) and \( T^+ \) in \( T_h \) such that \( F = T^- \cap T^+ \). Let \( n_F \) be the unit normal vector to \( F \) pointing from \( T^- \) towards \( T^+ \). For a double-valued function \( v \) on \( F \), its jump is defined as \( [v] = v^- - v^+ \) with \( v^\pm = v|_{T^\pm} \). Choosing nonnegative weights \( \omega_{T^-,F} \) and \( \omega_{T^+,F} \) such that \( \omega_{T^-,F} + \omega_{T^+,F} = 1 \), the weighted average of \( v \) on \( F \) is \( [v]_ω = \omega_{T^-,F} v^- + \omega_{T^+,F} v^+ \). The usual average consists of taking \( \omega_{T^-,F} = \omega_{T^+,F} = \frac{1}{2} \). When the diffusion tensor is strongly heterogeneous, it is better [4,9] to consider diffusion-dependent weights defined as \( \omega_{T^+,F} = (\delta_{K,F+} + \delta_{K,F-})^{-1} \delta_{K,F-} \) and \( \omega_{T^-,F} = (\delta_{K,F+} + \delta_{K,F-})^{-1} \delta_{K,F+} \) where \( \delta_{K,F±} = n_F^\prime(K[T±])n_F \). On boundary faces, we set \( [v] = v, [v]_ω = v, \omega_{T,F} = 1 \) (where \( T \) is the mesh element of which \( F \) is a face), and \( \delta_{K,F} = n_F'(K[T])n_F \) where \( n_F \) coincides with the outward unit normal of \( \Omega \).

Let \( k \geq 1 \). The dG approximation consists of finding \( u_h \in V^h_k \) such that \( B_h(u_h, v_h) = (f, v_h)_0,\Omega \) for all \( v_h \in V^h_k \) with the bilinear form

\[
B_h(v, w) = \sum_{T \in T_h} (K \nabla v, \nabla w)_0,T + \sum_{F \in F_h} \alpha h_F^{-1} \gamma_{K,F}([v], [w])_{0,F} - \sum_{F \in F_b} ([n_F'(K \nabla v)]_\omega, [w])_{0,F} + \theta(n_F'(K \nabla v)_\omega, [w])_{0,F}.
\]

The penalty coefficient \( \gamma_{K,F} \) is defined on interior faces as \( \gamma_{K,F} = (\delta_{K,F+} + \delta_{K,F-})^{-1} \delta_{K,F+} \delta_{K,F-} \) (i.e., it depends on the diffusion tensor via the harmonic average of the normal diffusivity) and as \( \gamma_{K,F} = \delta_{K,F} \) on boundary faces.
Furthermore, \( h_F \) denotes the diameter of \( F \), \( \alpha \) is a positive parameter, and \( \theta \) can take values in \( \{-1, 0, +1\} \). As usual with IP-like methods, if \( \theta \neq -1 \), the parameter \( \alpha \) must be chosen large enough to ensure that the bilinear form \( B_h \) is coercive. The threshold depends on the shape-regularity of the mesh family and the polynomial degree \( k \), but not on the meshsize and the diffusion tensor if the penalty parameter is designed as above. An optimal (with respect to meshsize) a priori error estimate is proven in [9] in the discrete energy-norm

\[
\| u \|_{T}^{2} = \sum_{T \in \mathcal{T}_{h}} \| u \|_{T}^{2}, \quad \| v \|_{T}^{2} = (K \nabla v, \nabla v)_{0,T} + \sum_{F \subset \partial T} \alpha h_{F}^{-1} \gamma_{K,F} \| u \|_{0,F},
\]

(4)

The estimate is robust with respect to diffusion anisotropies and only depends on local diffusion anisotropies.

3. The accurate \( H(\text{div}) \) flux reconstruction

Consider the Raviart–Thomas spaces of vector functions \( \mathbf{RT}_{h}^{k} = \{ v_{h} \in \mathbf{H}(\text{div}, \Omega); v_{h} |_{T} \in \mathbf{RT}_{T}^{k} \forall T \in \mathcal{T}_{h} \} \) where \( \mathbf{RT}_{T}^{k} = \mathbb{P}_{k}^{d}(T) + x \mathbb{P}_{k}(T) \). The reconstructed flux introduced in this Note, \( t_{h} \in \mathbf{RT}_{h}^{k} \), is specified through its natural degrees of freedom, namely for all \( F \in \mathcal{F}_{h} \) and \( q_{h} \in \mathbb{P}_{k}(F) \),

\[
(t_{h} \cdot n_{F}, q_{h})_{0,F} = - (n_{F} \cdot (K \nabla u_{h})_{\omega} + \alpha h_{F}^{-1} \gamma_{K,F} \| u_{h} \|, q_{h})_{0,F},
\]

(5)

and for all \( T \in \mathcal{T}_{h} \) and \( r_{h} \in \mathbb{P}_{k-1}(T) \),

\[
(t_{h}, r_{h})_{0,T} = -(K \nabla u_{h}, r_{h})_{0,T} + \theta \sum_{F \subset \partial T} \omega_{T,F}(n_{F} \cdot (K \nabla u_{h})_{\omega})_{0,F},
\]

(6)

**Theorem 3.1.** There holds \( \nabla \cdot t_{h} = \Pi_{h}^{k} f \).  

**Proof.** For all \( T \in \mathcal{T}_{h} \) and \( \xi \in \mathbb{P}_{k}(T) \), \( (f, \xi)_{0,T} = B_{h}(u_{h}, \xi \times 1)_{T} = -(t_{h}, \nabla \xi)_{0,T} + \sum_{F \subset \partial T} (t_{h} \cdot n_{T}, \xi)_{0,F} = (\nabla \cdot t_{h}, \xi)_{0,T} \) owing to (3), (5), and (6). □

The following result estimates the energy-norm of the error in the diffusive flux in terms of the discrete energy-norm of the primal error \( (u - u_{h}) \). In the sequel, \( A \lesssim B \) denotes the inequality \( A \leq c B \) with \( c \) independent of meshsize and of \( K \).

**Theorem 3.2.** There holds \( \| K^{1 \over 2} \nabla u + K^{-1 \over 2} t_{h} \|_{0,\Omega} \lesssim \max_{T \in \mathcal{T}_{h}} (A_{K,T} / \lambda_{K,T}) \| u - u_{h} \|_{\Omega} \).

**Proof.** Clearly, it suffices to estimate \( \| K^{1 \over 2} \nabla u_{h} + K^{-1 \over 2} t_{h} \|_{0,\Omega} \). Using scaling arguments and the Piola transformation, one first shows that for all \( T \in \mathcal{T}_{h} \) and \( v_{h} \in \mathbf{RT}_{h}^{k} \),

\[
\| v_{h} \|_{0,T}^{2} \lesssim h_{T} \sum_{F \subset \partial T} \| v_{h} \cdot n_{F} \|_{0,F}^{2} + \| \Pi_{h}^{k-1} v_{h} \|_{0,T}^{2}.
\]

(7)

We apply this estimate to \( v_{h} = (K \nabla u_{h} + t_{h}) |_{T} \in \mathbf{RT}_{h}^{k} \). Owing to (5)–(6) and using inverse inequalities,

\[
v_{h} \cdot n_{F} = \tilde{\omega}_{T,F} n_{F} \| (K \nabla u_{h}) \| + \alpha h_{F}^{-1} \gamma_{K,F} \| u_{h} \|
\]

and

\[
\| \Pi_{h}^{k-1} v_{h} \|_{0,T} \lesssim |\theta| h_{T}^{-1} A_{K,T} \sum_{F \subset \partial T} \omega_{T,F} \| u_{h} \|_{0,F},
\]

where \( \tilde{\omega}_{T,F} := 1 - \omega_{T,F} \), so that

\[
\lambda_{K,T}^{-1} \| v_{h} \|_{0,T}^{2} \lesssim \sum_{F \subset \partial T} \lambda_{K,T}^{-1} h_{T} \tilde{\omega}_{T,F} \| n_{F} \| (K \nabla u_{h}) \|_{0,F}^{2} + \sum_{F \subset \partial T} \lambda_{K,T}^{-1} \gamma_{K,F}^{-1} \| n_{F} \| \alpha_{K,T} \omega_{T,F} \| u_{h} \|_{0,F}^{2}.
\]

(8)

Let \( X \) and \( Y \) denote the two terms in the right-hand side. The first term is bounded using bubble functions, similarly to the a posteriori analysis of conforming finite elements; see [7] for details. The result is

\[
X \lesssim \max_{T \in \Delta_{f}} (A_{K,T} / \lambda_{K,T})^{2} \sum_{T' \in \Delta_{T}} \| u - u_{h} \|_{T'}^{2}.
\]
where $\Delta_T$ denotes the set of mesh elements sharing at least a face with $T$. The second term is bounded observing that
\[
\lambda_{K,T}^{-1}K^{2}_{T,F} \lesssim (A_{K,T}/\lambda_{K,T})\gamma_{K,F}^{2} \quad \text{and} \quad \lambda_{K,T}^{-1}K^{2}_{T,F} \omega_{T,F}^{2} \lesssim (A_{K,T}/\lambda_{K,T})^{2}\gamma_{K,F},
\]
leading to
\[
Y \lesssim (A_{K,T}/\lambda_{K,T})^{2} \sum_{T' \in \Delta_T} \|u - u_h\|_{T'}^{2}.
\]
Finally, summing over the mesh elements yields the desired result. \(\square\)

Because of the a priori estimate for $\|u - u_h\|_{\Omega}$ established in [9], Theorem 3.2 implies the same bound for the reconstructed flux $t_h$. A further consequence is that the computable quantity $\|K^{1/2}\nabla u_h + K^{-1/2}t_h\|_{0,\Omega}$ is optimal for the purpose of a posteriori error estimation; see [5,8] for details including numerical experiments.

References