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Probability Theory

On one property of Derrida–Ruelle cascades

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Abstract

We present new results about Poisson–Dirichlet point processes and Derrida–Ruelle cascades that are motivated by applications to mean-field spin glass models and, in particular, by the attempt to express Guerra's interpolation in the Sherrington–Kirkpatrick model entirely in the language of the cascades. ***To cite this article: D. Panchenko, M. Talagrand, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

Sur une propriété des cascades de Derrida–Ruelle. Nous présentons une nouvelle propriété des processus ponctuels de Poisson–Dirichlet et des cascades de Derrida–Ruelle. Cela nous permet d'exprimer l'interpolation de Guerra dans le modèle de verres de spin de Sherrington–Kirkpatric entièrement dans le langage des cascades. ***Pour citer cet article : D. Panchenko, M. Talagrand, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

1. Introduction and main results

Given $0 < m < 1$, consider a Poisson point process Π on $(0, +\infty)$ with intensity measure

$$d\mu_m(x) = x^{-1-m}\,dx \quad \text{for } x \in (0, +\infty). \tag{1}$$

This classical object in the theory of Poisson point processes plays a very important role in the mean-field spin glass models starting, probably, with the work of Ruelle in [11] that gives a mathematical foundation to Derrida's random energy model (REM) in [5] and its generalization (GREM) in [6]. A more detailed mathematical description of the REM can be found, for example, in [13], Chapter 1, and we refer to [4] and references therein for results about the GREM. A new description of the Derrida–Ruelle probability cascades and related results motivated by the physicist's cavity method were given in [3].

A breakthrough in the study of the Sherrington–Kirkpatrick model of spinglasses occurred with the introduction of a new interpolation by Guerra [7], which soon lead to the proof of the celebrated Parisi formula by the second author

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in [14]. It was suggested in [2] that Derrida–Ruelle cascades can be used to give an alternative (and in some sense, simpler) proof of the main bound in [7]. Another application in the same spirit was given in [9] where the analogue of Guerra's result was obtained for diluted mean-field models.

The arguments of [14] use not only the bound of Guerra [7] but crucially the precise form of the 'error terms' in the interpolation. The present work was motivated by an attempt to express these error terms in Guerra's interpolation (and the entire proof of [14]) completely in the language of Derrida–Ruelle cascades. (For more on that see upcoming second edition of [13]). It turned out that new properties of Derrida–Ruelle cascades are needed to achieve this goal and these are presented below. We will not discuss Guerra's interpolation here and refer the reader to [10] for details.

After the paper [10] was written, we found out that similar results appeared earlier in [1]. However, there seems to be a gap in the argument given in [1]. It seems impossible to dispense with Lemma 2.1 below which is not proved there.

2. Poisson–Dirichlet point process

For simplicity of notations, we will identify a process Π with its nondecreasing enumeration $(u_n)_{n\geqslant 1}$. Let S be a complete separable metric space that we will also view as a measurable space with Borel σ-algebra. Consider an i.i.d. sequence (X_n, Y_n) with distribution ν on $\mathbb{R}\times S$ independent of (u_n) and such that $X_n > 0$. Let ν_1, ν_2 denote the marginals of ν and ν_x denote a regular conditional distribution of Y given $X = x$. Suppose that $\mathbb{E}X < \infty$ and let ν_m be a probability measure on S defined by

$$\nu_m(B) = \int \frac{x^m}{\mathbb{E}X^m}\nu_x(B)\,d\nu_1(x), \tag{2}$$

which is obviously a distribution of Y under the change of density $X^m/\mathbb{E}X^m$, i.e. for any measurable real valued function ϕ,

$$\int \phi(y)\,d\nu_m(y) = \frac{\mathbb{E}X^m\phi(Y)}{\mathbb{E}X^m}.$$

The following holds:

Lemma 2.1. *Poisson point process (u_nX_n, Y_n) has the same distribution as a point process*

$$\big((\mathbb{E}X^m)^{1/m}u_n, Y_n'\big) \tag{3}$$

where (Y_n') is an i.i.d. sequence independent of (u_n) with distribution ν_m.

Remark. The special case where $Y_n = X_n$ appeared in [12]. Even though the proof of Lemma 2.1 is rather simple, the idea itself is nontrivial and the formulation seems to be new.

Proof. By the marking theorem [8], a point process (u_n, X_n, Y_n) is a Poisson point process with intensity measure $\mu\otimes\nu$ on $(0,\infty)\times(0,\infty)\times S$. By the mapping theorem [8], (u_nX_n, Y_n) is a Poisson point process with intensity measure given by the image of $\mu\otimes\nu$ under the mapping $(u,x,y)\mapsto(ux,y)$ whenever this measure has no atoms. Let us compute this image measure. Given two measurable sets $A\subseteq(0,\infty)$ and $B\subseteq S$,

$$\mu\otimes\nu(ux\in A, y\in B) = \int \mu(u: ux\in A)\nu_x(B)\,d\nu_1(x).$$

For $x > 0$ we have

$$\mu(u: xu\in A) = \int I(xu\in A)x^{-1-m}\,dx = u^m\int I(z\in A)z^{-1-m}\,dz = u^m\mu(A)$$

and, therefore,

$$\mu\otimes\nu(ux\in A, y\in B) = \int x^m\mu(A)\nu_x(B)\,d\nu_1(x) = \mathbb{E}X^m\mu(A)\otimes\nu_m(B).$$

Since the measure $\mathbb{E}X^m\mu$ is the intensity measure of a Poisson point process $((\mathbb{E}X^m)^{1/m}u_n)$ this finishes the proof. $\square$

3. Derrida–Ruelle cascades

Let us consider a sequence of parameters $0 < m_1 < m_2 < \cdots < m_k < 1$. We start by constructing a family of point processes on $(0, +\infty)$ as follows.

- Let $(u_{n_1})_{n_1 \geqslant 1}$ be a Poisson point process with intensity measure μ_{m_1} defined in (1).
- Recursively for $2 \leqslant l \leqslant k$, for all $(n_1, \ldots, n_{l-1}) \in \mathbb{N}^{l-1}$ we define independent Poisson point processes $(u_{n_1 \ldots n_{l-1} n_l})_{n_l \geqslant 1}$ with intensity measure μ_{m_l} independent of all previously constructed processes $(u_{n_1 \ldots n_j})$ for $j \leqslant l-1$.

Consider complete separable metric spaces $S_1, \ldots, S_k$ which we also view as measurable spaces with Borel σ-algebras and for $l \leqslant k$ let $S^l = S_1 \times \cdots \times S_l$. Consider a probability measure ν on S_1 and for $1 \leqslant l < k$ consider regular conditional distributions

$$\nu_l(\cdot|x) \quad \text{on } S_{l+1} \text{ for } x \in S^l. \tag{4}$$

We generate a process $Z_\alpha \in S^k$ indexed by $\alpha \in \mathbb{N}^k$ according to the following recursive procedure, that is similar to the procedure that generates the processes $(u_{n_1 \ldots n_{l-1} n_l})_{n_l \geqslant 1}$.

- Generate i.i.d. random variables $(z_{n_1})_{n_1 \geqslant 1}$ with distribution ν.
- Recursively over $2 \leqslant l \leqslant k$, given $(z_{n_1}, \ldots, z_{n_1 \ldots n_{l-1}})$ for all $n_1 \ldots n_{l-1} \in \mathbb{N}$, we generate i.i.d. sequences $(z_{n_1 \ldots n_{l-1} n_l})_{n_l \geqslant 1}$ with distributions
$$\nu_l(\cdot|z_{n_1}, \ldots, z_{n_1 \ldots n_{l-1}}) \tag{5}$$
independently for all $n_1, \ldots, n_{l-1}$.

Then, for each $\alpha = (n_1, \ldots, n_k)$ we define $Z_\alpha = (z_{n_1}, z_{n_1 n_2}, \ldots, z_{n_1 \ldots n_k})$. Given $\alpha \in \mathbb{N}^k$ we write $\alpha^l = (n_1, \ldots, n_l)$ and denote

$$u_{\alpha^l} = u_{n_1 \ldots n_l}, \quad z_{\alpha^l} = z_{n_1 \ldots n_l}, \quad \text{and} \quad Z_{\alpha^l} = (z_{n_1}, \ldots, z_{n_1 \ldots n_l}).$$

Consider a measurable function $X : S^k \to \mathbb{R}$ such that $\mathbb{E} \exp X(Z_\alpha) < \infty$. Let $X_\alpha = X(Z_\alpha)$ and recursively for $1 \leqslant l \leqslant k$ define

$$X_{\alpha^{l-1}} = \frac{1}{m_l} \log \mathbb{E}_l \exp m_l X_{\alpha^l} \quad \text{and} \quad W_{\alpha^l} = \exp m_l (X_{\alpha^l} - X_{\alpha^{l-1}}), \tag{6}$$

where $\mathbb{E}_l$ denotes the expectation conditionally on $(Z_{\alpha^{l-1}})_{\alpha \in \mathbb{N}^k}$. Thus, both X_{α^l} and W_{α^l} are functions of Z_{α^l}. In particular, $X_0 := X_{\alpha^0}$ is a constant. We can think of W_{α^l} as a function of two variables, $W_{\alpha^l} = W_l(Z_{\alpha^{l-1}}, z_{\alpha^l})$. Let us generate another process Z'_α exactly the same way as Z_α only instead of (5) we let

$$W_l(Z'_{\alpha^{l-1}}, \cdot)\, d\nu_l(\cdot|Z'_{\alpha^{l-1}}) \tag{7}$$

be the distribution of $(z'_{n_1 \ldots n_{l-1} n_l})_{n_l \geqslant 1}$ conditionally on $Z'_{\alpha^{l-1}}$. This is a probability measure because by (6) and (5)

$$\int W_l(Z'_{\alpha^{l-1}}, x)\, d\nu_l(x|Z'_{\alpha^{l-1}}) = \mathbb{E}_l \exp m_l (X_{\alpha^l} - X_{\alpha^{l-1}}) = 1.$$

The following generalizes Lemma 2.1 to the case of Derrida–Ruelle cascades:

Lemma 3.1. *If we define $e_{\alpha^l} = \exp(X_{\alpha^l} - X_{\alpha^{l-1}})$ then the point processes*

$$(u_{\alpha^1} e_{\alpha^1}, \ldots, u_{\alpha^k} e_{\alpha^k}, Z_{\alpha^k}) \quad \textit{and} \quad (u_{\alpha^1}, \ldots, u_{\alpha^k}, Z'_{\alpha^k}) \tag{8}$$

on ${\mathbb{R}^+}^k \times S^k$ have the same distribution.

The proof by induction using Lemma 2.1 can be found in [10]. As a consequence, defining $v_\alpha = \prod_{l \leqslant k} u_{n_1 \ldots n_l}$ and writing $v_\alpha \exp(X_\alpha - X_0) = \prod_{1 \leqslant l \leqslant k} u_{\alpha^l} e_{\alpha^l}$, we obtain the following:

Corollary 3.2. *The point processes*

$$\big(v_\alpha \exp(X_\alpha - X_0), Z_\alpha\big) \quad \textit{and} \quad (v_\alpha, Z'_\alpha) \tag{9}$$

have the same distribution.

This is the main tool to give an explicit analytic expression for the error terms in Guerra's interpolation when it is written via Derrida–Ruelle cascades. We refer the reader to [10] for the details.

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