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C. R. Acad. Sci. Paris, Ser. I 345 (2007) 673–677

COMPTES RENDUS MATHEMATIQUE

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Partial Differential Equations/Harmonic Analysis

Uncertainty principle and regularity for Boltzmann type equations

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Received 20 September 2007; accepted 16 October 2007

Available online 26 November 2007

Presented by Jean-Michel Bony

Abstract

We give a generalized version of uncertainty principle, and apply it to the study of regularization properties of solutions to kinetic equations. In particular, both linearized and nonlinear space inhomogeneous Boltzmann equations without Grad's cutoff assumption are considered. ***To cite this article: R. Alexandre et al., C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

Principe d'incertitude et régularité pour des équations de type Boltzmann. Nous montrons une version généralisée du principe d'incertitude, et l'appliquons à l'étude de propriétés de régularisation de solutions d'équations cinétiques. En particulier, nous considérons les versions linéarisée et non linéaire de l'équation de Boltzmann, sans faire l'hypothèse de troncature angulaire de Grad. ***Pour citer cet article : R. Alexandre et al., C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Version française abrégée

En vue d'étudier les propriétés d'hypoellipticité des équations cinétiques de type Boltzmann, dans le cadre des noyaux singuliers, c'est-à-dire sans faire l'hypothèse de troncature angulaire de Grad, nous prouvons tout d'abord une version adaptée du principe d'incertitude de Feffermann.

Le cadre de ce principe est le suivant. On considère deux fonctions positives a^+ et a^- de la variable $v \in \mathbb{R}^n$, avec $a^- \in L^\infty(\mathbb{R}^n)$. Pour $0 < s < 1$, on associe à l'opérateur $a_s(D_v) = |D_v|^{2s}$, la fonction $\tilde{a} = \tilde{a}(r) = r^{-2s}$ de la variable réelle $r > 0$. Pour un cube $Q \subset \mathbb{R}^n$ donné, de longueur de côté $l(Q)$, on désigne par Q^* tout cube tel que $Q \subset Q^*$ et $l(Q^*) = 2l(Q)$. Pour un tel couple de cubes (Q, Q^*), on leur associe le sous-ensemble suivant de $\mathbb{R}^n$

$$E(Q, Q^*) \equiv \{v \in Q^*,\ a^+(v) \geqslant \|a^-\|_{L^\infty(Q)} - \tilde{a}(l(Q))\}.$$

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1631-073X/$ – see front matter
doi:10.1016/j.crma.2007.10.032

Notre hypothèse principale est alors la suivante : il existe une constante $\kappa > 0$ telle que

$$\inf_{(Q,Q^*),\ l(Q^*)=2l(Q)} \frac{|E(Q,Q^*)|}{|Q^*|} \geqslant \kappa. \tag{H}$$

On a alors le théorème suivant :

Théorème 0.1. *Avec les notations et hypothèses précédentes, il existe une constante $C = C(\kappa, n)$ telle que*

$$\int_{\mathbb{R}^n} a^-(v)\big|f(v)\big|^2\,\mathrm{d}v \leqslant C \int_{\mathbb{R}^n} \big\{\big|a_s^{1/2}(D_v)f(v)\big|^2 + a^+(v)\big|f(v)\big|^2\big\}\,\mathrm{d}v,$$

pour toute fonction $f \in \mathcal{S}(\mathbb{R}^n)$.

A quelques modifications près, la même inégalité reste valable si l'opérateur $a_s^{1/2}$ est remplacé par l'opérateur $a_{s,\log}^{1/2} \equiv (\log\langle D_v\rangle)^s$. Comme application de ce principe, nous considérons le cadre des équations cinétiques, et nous nous limitons ici à un seul exemple, renvoyant à la version anglaise pour d'autres cas, et en particulier les équations de Boltzmann. On a alors le théorème suivant :

Théorème 0.2. *Soit $g \in H^{-s'}(\mathbb{R}^{2n+1})$ pour un certain $0 \leqslant s' < 1$, et soit $f \in L^2(\mathbb{R}^{2n+1})$ satisfaisant*

$$\partial_t f + v\cdot\nabla_x f = g \quad \text{in } \mathcal{D}'\big(\mathbb{R}^{2n+1}\big),$$

où $(t,x,v) \in \mathbb{R}^{1+n+n}$. On suppose de plus que $a_s^{1/2}(D_v)f \in L^2(\mathbb{R}^{2n+1})$, $0 < s \leqslant 1$. Alors

$$\frac{\langle D_x\rangle^{s(1-s')/(s+1)}}{(1+|v|^2)^{ss'/2(s+1)}} f \ \textit{et}\ \frac{\langle D_t\rangle^{s(1-s')/(s+1)}}{(1+|v|^2)^{s/2(s+1)}} f \in L^2\big(\mathbb{R}^{2n+1}\big).$$

Pour plus de détails et d'autres résultats, nous renvoyons à la version anglaise. Tous les résultats présentés ici, ainsi que des extensions, seront détaillés dans un article à paraitre [2].

1. Introduction and main results

Having in mind the study of regularity of weak solutions to Boltzmann type kinetic equations, by using tools from pseudo-differential operators theory, and more generally harmonic analysis, we shall first show a generalized version of Fefferman's uncertainty principle. For further details on this principle, we refer to [4–8]. Let us note that another method is still available, in some cases, see [10].

For variable $v \in \mathbb{R}^n$, let $a^+(v)$ and $a^-(v)$ be two nonnegative functions, with $a^- \in L^\infty(\mathbb{R}^n)$. Let $a_s(D_v) \equiv |D_v|^{2s}$ for $0 < s < 1$. We associate, for variable $r > 0$, the function $\tilde{a}(r) = r^{-2s}$.

For any cube $Q \subset \mathbb{R}^n$, we denote by $l(Q)$ its side length, and by Q^* any cube such that $Q \subset Q^*$ with $l(Q^*) = 2l(Q)$. A given couple of cubes (Q, Q^*) is then associated with the subset

$$E(Q,Q^*) = \big\{v \in Q^*, a^+(v) \geqslant \|a^-\|_{L^\infty(Q)} - \tilde{a}\big(l(Q)\big)\big\}.$$

Our main assumption on the functions a^+ and a^- is then the following: we assume there exists a constant $\kappa > 0$ such that

$$\inf_{(Q,Q^*),\ l(Q^*)=2l(Q)} \frac{|E(Q,Q^*)|}{|Q^*|} \geqslant \kappa. \tag{H}$$

Theorem 1.1 *(Uncertainty principle). Under the above notations and assumptions, in particular* (H)*, there exists a constant $C = C(\kappa, n)$, such that for all $f \in \mathcal{S}(\mathbb{R}^n)$, one has*

$$\int_{\mathbb{R}^n} a^-(v)\big|f(v)\big|^2\,\mathrm{d}v \leqslant C \int_{\mathbb{R}^n} \big\{\big|a_s^{1/2}(D_v)f(v)\big|^2 + a^+(v)\big|f(v)\big|^2\big\}\,\mathrm{d}v.$$

A similar estimate holds also true, if the operator $a_s^{1/2}$ is replaced by $a_{s,\log}^{1/2} \equiv (\log\langle D_v\rangle)^s$ with $s > \frac{1}{2}$.

Our main interest lies in the application of this generalized uncertainty principle to the study of hypoellipticity of kinetic equations. To compare our approach with [3], a typical example is given by the following result:

Theorem 1.2. *Assume that $g \in H^{-s'}(\mathbb{R}^{2n+1})$, for some $0 \leqslant s' < 1$ and $f \in L^2(\mathbb{R}^{2n+1})$ is a weak solution of the following kinetic equation*

$$\partial_t f + v \cdot \partial_x f = g,$$

where $(t, x, v) \in \mathbb{R}^{1+n+n}$. If furthermore $a_s^{1/2}(D_v) f \in L^2(\mathbb{R}^{2n+1})$, $0 < s \leqslant 1$, then it follows that

$$\frac{\langle D_x \rangle^{s(1-s')/(s+1)} f}{(1+|v|^2)^{ss'/2(s+1)}} \text{ and } \frac{\langle D_t \rangle^{s(1-s')/(s+1)} f}{(1+|v|^2)^{s/2(s+1)}} \in L^2(\mathbb{R}^{2n+1}).$$

As a first step to a regularity result for the full space inhomogeneous nonlinear Boltzmann equation, we now consider applications of the above hypoellipticity estimation and method to the linearized Boltzmann equation, in the case of non cutoff cross-sections. For further details on Boltzmann equation in this case, we refer to [11].

We shall work in dimension $n = 3$, though other dimensions could be considered as well. The usual Boltzmann quadratic operator $Q(g, f)$ is given by:

$$Q(g, f) = \iint_{\mathbb{R}^3 \mathbb{S}^2} B(v - v_*, \sigma)\big\{g(v'_*) f(v') - g(v_*) f(v)\big\}\, d\sigma\, dv_*.$$

Above, σ-representation has been used for describing the post and pre collisional velocities. Furthermore, we choose the collision kernel B to be of Maxwellian molecule type, and corresponding to inverse power laws type potentials with small singularity. That is, we assume, for some constant $K > 0$ and some $0 < s < \frac{1}{2}$

$$B = b(\cos\theta) \sim K|\theta|^{-2-2s}, \quad \text{when } \theta \to 0, \tag{1}$$

so that Grad's cutoff assumption is not satisfied. Note that for the Maxwellian molecules, $s = 1/4$. The singularity of collision kernel (1) implies the following subelliptic estimate: If $g \in L^1_2 \cap L \log L(\mathbb{R}^3_v)$ and $g \geqslant 0$, then there exists a constant $C_g > 0$ such that

$$C_g^{-1} \big\| a_s^{1/2}(D_v) f \big\|^2_{L^2(\mathbb{R}^3_v)} \leqslant \big(-Q(g, f), f\big)_{L^2(\mathbb{R}^3_v)} + C_g \|f\|^2_{L^2(\mathbb{R}^3_v)},$$

for any smooth function $f \in H^2(\mathbb{R}^3_v)$. This subelliptic estimate was crucially used in the proof of the regularizing effects of the cross-sections on the solutions for the spatially homogeneous Boltzmann equations, cf. [1,9] and the references therein. In this work, we apply it to the spatially inhomogeneous problem through uncertainty principle.

The linearized Boltzmann operator around the normalized Maxwellian distribution $\mu(v) = (2\pi)^{-3/2} e^{-|v|^2/2}$ is then given by $\mathcal{L} f = Q(\mu, f) + Q(f, \mu)$, along with the following Cauchy problem

$$\partial_t f + v \cdot \nabla_x f = \mathcal{L} f, \quad x,\ v \in \mathbb{R}^3,\ t > 0; \quad f|_{t=0} = f_0. \tag{2}$$

Theorem 1.3. *Assume that $\langle v \rangle^l f_0 \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$ for all $l \in \mathbb{N}$. Then there exists a unique weak solution f of Cauchy problem* (2) *such that*

$$\langle v \rangle^l f \in L^\infty \cap L^2\big(]0, T[\times \mathbb{R}^6\big), \quad \forall l \in \mathbb{N},\ \forall T > 0.$$

Furthermore, for all $l \in \mathbb{N}$, for all $0 < \delta < T' < T$, it follows that

$$\langle v \rangle^l f \in H^{+\infty}\big(]\delta, T'[\times \mathbb{R}^6\big).$$

As regard to the full nonlinear space inhomogeneous Boltzmann equation

$$\partial_t f + v \cdot \nabla_x f = Q(f, f), \quad x, v \in \mathbb{R}^3,\ t > 0. \tag{3}$$

One has the following:

Theorem 1.4. *Assume that* $f \in L^\infty(]0,T[; L^1_2 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3))$ *is a weak solution of Eq.* (3), *and*

$$\big(1+|D_v|^2\big)^{s/2} f \in L^2_{\mathrm{loc}}\big(]0,T[\times\mathbb{R}^6\big), \qquad \big(1+|D_v|^2\big)^{-s} Q(f,f) \in L^2\big(]0,T[\times\mathbb{R}^6\big).$$

Then, it follows that

$$\big(1+|D_t|^2+|D_x|^2\big)^{\frac{s(1-2s)}{2(s+1)}} f \in L^2_{\mathrm{loc}}\big(]0,T[\times\mathbb{R}^6\big).$$

Note carefully that in the above result, we have assumed the existence of weak solutions in the class of functions mentioned in the assumptions. Such an existence theorem is still not available at present, and it is currently under study. Further details of the above results will be given in [2].

2. Ideas of the proofs

2.1. Proof of Theorem 1.1

For $k \in \mathbb{Z}$, let $\mathcal{D}_k$ be the collection of cubes with side length 2^{-k} and vertices located at points in $2^{-k}\mathbb{Z}^n$, and $\mathcal{D} = \bigcup_{k\in\mathbb{Z}} \mathcal{D}_k$ the set of all dyadic cubes. If a cube $Q \in \mathcal{D}$ is obtained by division of a bigger dyadic cube Q' with double side length, we call Q' the mother cube of Q.

For a constant $A > 0$ and a function $a^- \in L^\infty(\mathbb{R}^n)$, a cube Q is said to satisfy property $\mathcal{F}(A, a^-)$ if

$$\|a^-\|_{L^\infty(Q)} \leqslant A\tilde{a}\big(l(Q)\big).$$

Then, we denote by $\mathcal{M}(A, a^-)$ the set of all dyadic cubes $Q \in \mathcal{D}$ satisfying $\mathcal{F}(A, a^-)$ but not their mother cube. Clearly, if a cube Q' satisfies $\mathcal{F}(A, a^-)$ then so does any cube $Q \subset Q'$. It follows that we have a nonoverlapping covering on $\mathbb{R}^n$ by the elements in $\mathcal{M}(A, a^-)$. For such a covering, another locally finite covering on $\mathbb{R}^n$ can be constructed, taking into account the hypothesis (H).

Proposition 2.1. *Assume the hypothesis* (H). *Then there exists a constant* A_s *depending only on* s*, such that for all* $A \geqslant A_s$*, for the collection of cubes* $Q_j \in \mathcal{M}(A, a^-)$*, there exists another collection of cubes* $\tilde{Q}_j$ *satisfying the following*:

(i) $Q_j \subset \tilde{Q}_j \subset Q'_j$, *where* Q'_j *denotes a mother cube of* Q_j;

(ii) *There exists a dual pair of cubes* (Q, Q^*), Q *being of side length* $\frac{1}{2}l(Q_j)$ *such that*

$$\big|\tilde{Q}_j \cap E_j(Q, Q^*)\big| \geqslant \frac{\kappa}{2}|Q_j|,$$

and

$$a^+(v) \geqslant C_1\tilde{a}\big(l(Q_j)\big) \geqslant C_2\|a^-\|_{L^\infty(Q_j)}, \quad v \in E_j(Q, Q^*),$$

where C_1, C_2 *are the constants independents of* j.

(iii) *The collection* $\{\tilde{Q}_j\}$ *is a locally finite covering of* $\mathbb{R}^n$, *and the number of overlapping is order* $\mathrm{O}(\kappa^{-n})$.

The proof of this result follows by a detailed checking case by case, and is detailed in [2]. Once this result available the final proof of Theorem 1.1 follows by using the equivalent integral type definition of Sobolev spaces.

2.2. Proof of Theorem 1.2

The following estimate is seeked for:

$$\big\|A(v, D_t, D_x)f\big\|^2_{L^2(\mathbb{R}^{2n+1})} \leqslant C\big\{\big\|a_s^{1/2}(D_v)f\big\|^2_{L^2(\mathbb{R}^{2n+1})} + \|X_0 f\|^2_{H^{-s'}(\mathbb{R}^{2n+1})} + \|f\|^2_{L^2(\mathbb{R}^{2n+1})}\big\},$$

for all $f \in \mathcal{S}(\mathbb{R}^{2n+1})$, where $X_0 \equiv (\partial_t + v\cdot\nabla_x)$ and

$$A(v, D_t, D_x)f = \frac{\langle D_x\rangle^{s(1-s')/(s+1)} f}{(1+|v|^2)^{ss'/2(s+1)}} + \frac{\langle D_t\rangle^{s(1-s')/(s+1)} f}{(1+|v|^2)^{s/2(s+1)}}.$$

We denote by τ, η and ξ the dual (Fourier) variables corresponding to t, x and v respectively. Choose $\chi(\tau,\eta,\xi)\in S^0(\mathbb{R}^{2n+1})$, such that $\chi=1$ on $\Gamma=\{(\tau,\eta,\xi)\in\mathbb{R}^{1+2n}_{\tau,\eta,\xi};\ |\tau|^2+|\eta|^2\leqslant 1+|\xi|^2/2\}$, and $\chi=0$ outside of $\tilde\Gamma=\{|\tau|^2+|\eta|^2\geqslant 1+|\xi|^2\}$. Denoting by $\chi(D)$ the pseudo-differential operator with symbol χ, it is enough to prove

$$\big\|A(v,D_t,D_x)\chi(D)f\big\|^2_{L^2(\mathbb{R}^{2n+1})}\leqslant C\big\{\big\|a_s^{1/2}(D_v)\chi(D)f\big\|^2_{L^2(\mathbb{R}^{2n+1})}+\big\|X_0\chi(D)f\big\|^2_{H^{-s'}(\mathbb{R}^{2n+1})}+\|f\|^2_{L^2(\mathbb{R}^{2n+1})}\big\}.$$

Since supp $\mathcal{F}_{t,x,v}(\chi(D)f)\subset\tilde\Gamma$, one deduces that

$$\int_{\mathbb{R}^n_v}\bigg(\iint_{\mathbb{R}^{n+1}_{\tau,\eta}}\frac{|\tau+v\cdot\eta|^2}{(1+\tau^2+|\eta|^2)^{s'}}\big|\mathcal{F}_{t,x}\big(\chi(D)f\big)(v,\tau,\eta)\big|^2\,\mathrm{d}\tau\,\mathrm{d}\eta\bigg)\,\mathrm{d}v\leqslant\big\|X_0\chi(D)f\big\|^2_{L^2(\mathbb{R}^{2n+1})}.$$

The operator $a_s^{1/2}(D_v)$ is independent of t,x, then it is enough to prove

$$c\big(a^-(v,\tau,\eta)u,u\big)_{L^2(\mathbb{R}^n)}\leqslant\mathrm{Re}\big(\big(a_s(D_v)+a^+(v,\tau,\eta)\big)u,u\big)_{L^2(\mathbb{R}^n)},\qquad u\in\mathcal{S}(\mathbb{R}^n),\tag{4}$$

for all $(\tau,\eta)\in\mathbb{R}^{1+n}$ such that $|\tau|,|\eta|\geqslant R_0$ for some big constant $R_0>0$, where for some $c_0>0$

$$a^-(v,\tau,\eta)=c_0\frac{|\tau|^{2s(1-s')/(s+1)}}{(1+|v|^2)^{s/(s+1)}}+c_0\frac{|\eta|^{2s(1-s')/(s+1)}}{(1+|v|^2)^{ss'/(s+1)}}=a_1^-(v,\tau)+a_2^-(v,\eta)$$

and

$$a^+(v,\tau,\eta)=1+\frac{|\tau+v\cdot\eta|^2}{(1+\tau^2+|\eta|^2)^{s'}}.$$

The final claim is that the condition (H) for the uncertainty principle in the form of (4) is true when $(\tau,\eta)\in\mathbb{R}^{1+n}$ is viewed as a parameter.

Finally, for Theorems 1.3 and 1.4, the existence and uniqueness of weak solution are obtained by approximation of equations with angular cutoff, and the regularity of weak solution is an application of Theorem 1.2 by using induction.

Acknowledgements

We would like to thank Tai-Ping Liu for his insightful discussion. The research of the second author was supported by Grant-in-Aid for Scientific Research No.18540213, Japan Society of the Promotion of Science. Third author's research is supported by Liu Bie Ju Centre for Mathematical Sciences and Department of Mathematics, City University of Hong Kong. Fifth author's research was supported by the RGC Competitive Earmarked Research Grant of Hong Kong, CityU#102606, and the Changjiang Scholar Program of Chinese Educational Ministry, Shanghai Jiao Tong University.

References

[1] R. Alexandre, Integral estimates for linear singular operator linked with Boltzmann operator. Part I: small singularities $0<\nu<1$, Indiana Univ. J. Math. 55–6 (2007).
[2] R. Alexandre, Y. Morimoto, S. Ukai, C.-Y. Xu, T. Yang, in preparation.
[3] F. Bouchut, Hypoelliptic regularity in kinetic equations, J. Math. Pure Appl. 81 (2002) 1135–1159.
[4] C. Fefferman, The uncertainty principle, Bull. Amer. Math. Soc. 9 (1983) 129–206.
[5] C. Fefferman, D.H. Phong, The uncertainty principle and sharp Gårding inequalities, Comm. Pure. Appl. Math. 34 (1981) 285–331.
[6] Y. Morimoto, The uncertainty principle and hypoelliptic operators, Publ. RIMS Kyoto Univ. 23 (1987) 955–964.
[7] Y. Morimoto, Estimates for degenerate Schrödinger operators and hypoellipticity for infinitely degenerate elliptic operators, J. Math. Kyoto Univ. 32 (1992) 333–372.
[8] Y. Morimoto, T. Morioka, The positivity of Schrödinger operators and the hypoellipticity of second order degenerate elliptic operators, Bull. Sci. Math. 121 (1997) 507–547.
[9] Y. Morimoto, S. Ukai, C.-J. Xu, T. Yang, Regularity of solutions to the spatially homogeneous Boltzmann equation without angular cutoff, preprint.
[10] Y. Morimoto, C.-J. Xu, Hypoellipticity for a class of kinetic equations, J. Math. Kyoto. Univ. 47 (2007), in press.
[11] C. Villani, A review of mathematical topics in collisional kinetic theory, in: S. Friedlander, D. Serre (Eds.), Handbook of Mathematical Fluid Dynamics, vol. I, North-Holland, Amsterdam, 2002, pp. 71–305.