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Partial Differential Equations/Mathematical Problems in Mechanics

Asymptotic stability of shock profiles in radiative hydrodynamics

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Abstract

In a former work, we have shown the existence of smooth shock profiles for a model of nonequilibrium radiative hydrodynamics. In this Note, we show that such shock profiles are asymptotically stable for zero mass perturbation. Following the case of viscous shock profiles, the analysis relies on energy estimates for the integrated system. ***To cite this article: C. Lin et al., C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

Stabilité asymptotique de profils de choc pour un modèle de transfert radiatif. Dans un précédent travail, nous avons montré l'existence de profils de choc pour un modèle de transfert radiatif hors-équilibre. Dans cette Note, nous montrons que de tels profils de choc sont asymptotiquement stables pour des perturbations de masse nulle. Comme dans le cas des profils de choc visqueux, l'analyse repose sur des estimations d'énergie pour le système intégré. ***Pour citer cet article : C. Lin et al., C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

1. Introduction and main result

In this Note, we consider a simple model that describes the interaction of an inviscid compressible fluid with photons. The fluid is described by its specific volume v, its velocity u, and its specific energy e. The photons are described by the radiation energy n. The evolution of the fluid is governed by the one-dimensional compressible Euler equations, that are coupled with an elliptic equation for the radiative energy n. In Lagrangian coordinates, the model reads:

$$\begin{cases} \partial_t v - \partial_x u = 0, \\ \partial_t u + \partial_x p = 0, \\ \partial_t (e + u^2/2) + \partial_x (pu + q) = 0, \\ -\partial_x \left(\dfrac{\partial_x q}{v} \right) + vq + \partial_x (\theta^4) = 0, \end{cases} \tag{1}$$

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where θ denotes the temperature of the fluid, p denotes the pressure, and q is the radiative flux, that is related to the radiative energy n through the relation $q = (\partial_x n)/v$. We assume that the fluid obeys the perfect gas pressure law:

$$p = \frac{R\theta}{v} = \frac{(\gamma - 1)e}{v}, \tag{2}$$

where R is the perfect gas constant, and γ is the ratio of specific heats at constant pressure c_p and constant volume c_v. (Recall that $c_p = \gamma R/(\gamma - 1)$ and $c_v = R/(\gamma - 1)$.) In [8], we have studied the existence of shock profiles solutions to (1), see also [2] for a preliminary analysis. Shock profiles are traveling waves solutions that connect asymptotic states defining a shock wave solution to the classical Euler equations (i.e. without radiation). More precisely, let:

$$(v, u, e) = \begin{cases} (v_+, u_+, e_+), & \text{if } x > st, \\ (v_-, u_-, e_-), & \text{if } x < st, \end{cases} \tag{3}$$

be a shock wave with speed s solution to the classical Euler equations without radiation:

$$\begin{cases} \partial_t v - \partial_x u = 0, \\ \partial_t u + \partial_x p = 0, \\ \partial_t \big(e + u^2/2\big) + \partial_x (pu) = 0. \end{cases} \tag{4}$$

In other words, $(v_\pm, u_\pm, e_\pm, s)$ satisfy the Rankine–Hugoniot jump conditions together with the classical entropy inequality, see e.g. [10]. Then a shock profile is a traveling wave solution $(v, u, e, q)(x - st)$ of (1), (2) such that:

$$\lim_{\xi \to \pm\infty} (v, u, e, q)(\xi) = (v_\pm, u_\pm, e_\pm, 0).$$

The existence of shock profiles has been proved in [8] for shock waves of small amplitude. Moreover, we have shown in [8] that as the amplitude of the shock decreases, the smoothness of the shock profile increases. In particular, one can obtain C^k shock profiles for any integer $k \geqslant 2$. The main result of this Note deals with the asymptotic stability of such shock profiles with respect to small initial perturbations, and can be stated as follows:

Theorem 1.1. *Let $\gamma \in]1, 2]$, $v_- > 0$, $u_- \in \mathbb{R}$, $e_- > 0$, and let (v_+, u_+, e_+) be such that* (3) *defines a shock wave with speed s for the Euler equations* (4)*. Then if (v_+, u_+, e_+) is sufficiently close to (v_-, u_-, e_-), there exists a C^4 shock profile $(V, U, E, Q)(x - st)$ connecting $(v_\pm, u_\pm, e_\pm, 0)$, and this shock profile is asymptotically stable with respect to zero mass perturbations. More precisely, if $(\Phi_0, \Psi_0, W_0) \in H^3(\mathbb{R})$ are sufficiently small, then the system* (1)–(2) *with initial data*:

$$\big(v, u, e + u^2/2\big)_{|_{t=0}} = \big(V, U, E + U^2/2\big) + (\Phi_0, \Psi_0, c_v W_0 + U\Psi_0)',$$

has a unique global in time smooth solution $(v, u, e, q)(t, x)$, and:

$$\lim_{t \to +\infty} \sup_{x \in \mathbb{R}} \big|(v, u, e, q)(t, x) - (V, U, E, Q)(x - st)\big| = 0.$$

Theorem 1.1 is nothing but the extension of the stability result of [5] to the system (1). In particular, the smoothness of the shock profile and the smoothness of the initial perturbations are the same in both cases. The model considered in [5] is a Burgers equation coupled with an elliptic equation for the radiation, so the equation is scalar while here we deal with a system. The perturbations are assumed to have zero mass in order to follow the strategy of [1] that was developed in the case of systems of conservation laws with artificial viscosity, see also [3] for the case of compressible Navier–Stokes equations. We refer to [9,11] for some physical background on radiative hydrodynamics. We also refer to [6] for general existence results of shock profiles for hyperbolic-elliptic systems. The asymptotic stability of shock profiles is still open in such a general context. Eventually, we recall that the asymptotic stability of constant states for (1) was established in [4] for general initial perturbations (that is, without zero mass assumption).

2. Asymptotic stability of shock profiles

In this section, we give an outline of the proof of Theorem 1.1. We follow the analysis of [3] and decompose the solution to (1) in the following way:

$$(v, u, e, q)(t, x) = (V, U, c_v\Theta, Q)(x - st) + (\phi, \psi, c_v\omega, z)(t, x - st).$$

We rewrite (1) for the unknowns ϕ, ψ, ω, z, that depend on the variables t and $\xi = x - st$, and we also make use of the zero mass assumption for the initial data. In particular, we can look for the solution ϕ, ψ, ω, z under the form:

$$(\phi, \psi, \omega, z) = \left(\partial_\xi \Phi, \partial_\xi \Psi, \partial_\xi W + \frac{1}{c_v}\left(U'\Psi - \frac{(\partial_\xi \Psi)^2}{2}\right), \frac{1}{V + \partial_\xi \Phi}\left(\partial_\xi Z - \left(\frac{Q'}{V} - \Theta^4\right)' \frac{\partial_\xi \Phi}{V}\right)\right). \tag{5}$$

The system satisfied by the perturbations ϕ, ψ, ω, z reads:

$$\begin{cases} \partial_t \phi - s\partial_\xi \phi - \partial_\xi \psi = 0, \\ \partial_t \psi - s\partial_\xi \psi + R\partial_\xi\left(\dfrac{\Theta + \omega}{V + \phi} - \dfrac{\Theta}{V}\right) = 0, \\ \partial_t(c_v\omega + U\psi + \psi^2/2) - s\partial_\xi(c_v\omega + U\psi + \psi^2/2) + R\partial_\xi\left(\dfrac{(\Theta + \omega)(U + \psi)}{V + \phi} - \dfrac{\Theta U}{V}\right) + \partial_\xi z = 0, \\ \dfrac{V + \phi}{V}\left(\dfrac{Q'}{V} - \Theta^4\right)' - \partial_\xi\left(\dfrac{Q' + \partial_\xi z}{V + \phi}\right) + (V + \phi)z + \partial_\xi(\Theta + \omega)^4 = 0, \end{cases} \tag{6}$$

while the system for the integrated perturbations Φ, Ψ, W, Z reads:

$$\begin{cases} \partial_t \Phi - s\partial_\xi \Phi - \partial_\xi \Psi = 0, \\ \partial_t \Psi - s\partial_\xi \Psi - \dfrac{R\Theta\partial_\xi \Phi}{V(V + \phi)} + \dfrac{R}{V + \phi}\left(\partial_\xi W + \dfrac{1}{c_v}(U'\Psi - \psi^2/2)\right) = 0, \\ c_v(\partial_t W - s\partial_\xi W) + R\dfrac{\Theta + \omega}{V + \phi}\partial_\xi \Psi - sU'\Psi + \dfrac{\partial_\xi Z}{V + \phi} - \dfrac{(Q'/V - \Theta^4)'\phi}{V(V + \phi)} = 0, \\ (V + \phi)Z - \partial_\xi\left(\dfrac{\partial_\xi Z}{V + \phi}\right) + \partial_\xi\left(\left(\dfrac{Q'}{V} - \Theta^4\right)' \dfrac{\phi}{V}\right) + \dfrac{Q'\phi}{V} + (V + \phi)\big((\Theta + \omega)^4 - \Theta^4\big) = 0. \end{cases} \tag{7}$$

The energy functional for (6)–(7) is defined as follows:

$$\begin{aligned} N(t)^2 := \sup_{0 \leqslant \tau \leqslant t} & \left\|(\Phi, \Psi, W)(\tau)\right\|^2_{L^2(\mathbb{R})} + \left\|(\phi, \psi, \omega)(\tau)\right\|^2_{H^2(\mathbb{R})} + \left\|z(\tau)\right\|^2_{H^3(\mathbb{R})} \\ & + \int_0^t \left\||V'|^{1/2}(\Psi, W)(\tau)\right\|^2_{L^2(\mathbb{R})} + \left\|Z(\tau)\right\|^2_{H^1(\mathbb{R})} + \left\|(\phi, \psi, \omega)(\tau)\right\|^2_{H^2(\mathbb{R})} + \left\|z(\tau)\right\|^2_{H^3(\mathbb{R})} \, \mathrm{d}\tau. \end{aligned} \tag{8}$$

This functional is inspired from [4] and [3]. Theorem 1.1 is then obtained by first proving the following result:

Proposition 2.1. *Let $\gamma \in \,]1, 2]$. Let $T > 0$, let $(\Phi, \Psi, W, Z) \in C([0, T]; H^3(\mathbb{R})^3 \times H^4(\mathbb{R}))$ solution to* (7)*, and let $(\phi, \psi, \omega, z) \in C([0, T]; H^2(\mathbb{R})^3 \times H^3(\mathbb{R}))$ be given by* (5)*. Then there exist two constants $\delta > 0$ and $C > 0$, such that if $N(T) \leqslant \delta$ and if the amplitude of the shock is sufficiently small, then one has*:

$$N(T)^2 \leqslant C\big(N(0)^2 + N(T)^3\big).$$

To show Proposition 2.1, one first derives an L^2 estimate for (7) by following [3]. The novelty relies in the elliptic equation for Z, which yields a few modifications in the symmetrization of the equations. One derives similarly L^2 estimates for (6). The main difference with respect to [3] is that we need to symmetrize the quasilinear form of (6), while in [3] the authors first linearized the equations around the shock profile solution and treated the linearization errors as source terms. This strategy cannot be followed because the dissipation introduced by the elliptic equation is not strong enough to control the linearization errors. However, the standard symmetrization of the quasilinear system (6) works, and the last ingredient in the proof is the H^1 estimate for the elliptic equation satisfied by z (which tells more or less that the H^3 norm of z is controlled by the H^2 norm of ω). The global existence of a smooth solution, as well as the asymptotic decay follows directly from Proposition 2.1, see e.g. [4]. Compared to the results of [1,3] obtained for the viscous regularization of hyperbolic conservation laws, we need here to consider initial integrated perturbations in H^3, and not only in H^2 as in [1,3]. This is due to the fact that the dissipation is not as strong as in the viscous case. In particular, we need to control the solution in $L^\infty_t(W^{1,\infty}_\xi)$, while in [1,3] the authors only needed a control in $L^2_t(W^{1,\infty}_\xi)$. This higher regularity is the main difference between coupled hyperbolic-elliptic systems and viscous regularizations of hyperbolic conservation laws. Detailed proofs of the above results may be found in [7].

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