On perfect fluids with bounded vorticity

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Abstract
This Note is devoted to studying the incompressible Euler equations. First, we prove global existence for three-dimensional axisymmetric solutions without swirl under a regularity assumption which is very close to the one which has been introduced in the two-dimensional setting by V. Yudovich (1963). Second, we state uniqueness in the general N-dimensional case for bounded solutions with bounded vorticity.

Résumé

1. Introduction
We are concerned with the Cauchy problem for the N-dimensional incompressible Euler equations:

$$\begin{align*}
\partial_t v + v \cdot \nabla v + \nabla p &= 0, \\
\text{div } v &= 0 \\
(t, x) &\in \mathbb{R} \times \mathbb{R}^N.
\end{align*}$$

(E)

In dimension two, global existence and uniqueness has been stated by V. Yudovich in [8] for data with bounded vorticity. The proof relies on the following facts: first, the vorticity \( \omega \) associated to \( v \) (which, in dimension two, reduces to a scalar function) is transported by the flow hence has constant \( L^\infty \) norm; second, having \( \omega \) bounded and \( \text{div } v = 0 \) implies that \( v \) is quasi-Lipschitz. One can then prove a stability estimate in energy norm by taking advantage of a generalized Gronwall lemma.

In dimension three, the problem of global solvability is much more involved. Indeed the vorticity (which may be identified with a solenoidal vector-field) is transported by the flow as a vector-field, namely

$$\partial_t \omega + v \cdot \nabla \omega = \omega \cdot \nabla v.$$
Hence the $L^\infty$ norm of $\omega$ may grow in time and the problem of global solvability has remained unsolved. We focus on the case of axisymmetric data without swirl: the initial velocity $v^0$ is assumed to be given in cylindrical coordinates $(r, \theta, z)$ by $v^0(r, \theta, z) = v_0^0(r, \theta) e_r + v_z^0(r, \theta) e_z$ where $e_r$ (resp. $e_z$) stands for the unit outer radial (resp. vertical) vector. The vorticity $\omega^0$ then reduces to

$$\omega^0(r, \theta, z) = \omega_\theta^0(r, \theta) e_\theta \quad \text{with} \quad \omega_\theta := \partial_r v_\theta - \partial_\theta v_r \quad \text{and} \quad e_\theta \times e_r.$$ 

Global well-posedness for (E) with axisymmetric data has been first proved by M. Ukhovskii and V. Yudovich in [6] under the additional assumption that $v^0 \in H^1$ and that the initial vorticity $\omega^0$ is in $L^\infty$ and satisfies $r^{-1} \omega^0 \in L^2 \cap L^\infty$ (see also [4]). The proof relies on the fact that the quantity $r^{-1} \omega_\theta$ is transported by the flow, hence plays the same role as the vorticity in dimension two. In terms of regularity in Sobolev spaces, [6]'s assumptions are stronger than those which are needed to have local well-posedness in dimension three. Indeed, local existence holds true in $H^s$ whenever $s > 5/2$ whereas $s > 7/2$ is required for having $r^{-1} \omega^0$ in $L^\infty$ for all $v^0 \in H^s(\mathbb{R}^3)$. This gap has been filled in by T. Shirota and T. Yanagisawa in [5].

In the present Note, we aim at getting a global existence result as close as possible to that of Yudovich in dimension two. As a matter of fact, we strive for a global result in a functional space (for the vorticity) which has the same scaling invariance as $L^\infty(\mathbb{R}^3)$. This is achieved in the following statement:

**Theorem 1.1.** Let $\omega^0$ be an axisymmetric function in $L^{3,1} \cap L^\infty$ such that $r^{-1} \omega^0 \in L^{3,1}$. Let $v^0$ be a bounded axisymmetric solenoidal vector-field with vorticity $\omega^0 e_\theta$. Then Euler equations (E) have a global solution $(v, \nabla p)$ with $v \in C_w(\mathbb{R}; C^1_\omega)$ and $\nabla p \in C(\mathbb{R}; C^{1-\varepsilon})$ for all $\varepsilon > 0$. Besides, the vorticity satisfies

$$\omega \in L^\infty_{loc}(\mathbb{R}; L^{3,1} \cap L^\infty) \quad \text{and} \quad \|r^{-1} \omega(t)\|_{L^{3,1}} = \|r^{-1} \omega^0\|_{L^{3,1}} \quad \text{for all} \ t \in \mathbb{R}.$$ 

Above, the notation $C^1_\omega$ stands for the Zygmund space of bounded continuous functions $f$ such that there exists a constant $A$ so that $|f(x + y) + f(x - y) - 2f(x)| \leq A|y|^2$ for all $(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3$, and $L^{3,1}$ is the Lorentz space which may be defined by real interpolation by $L^{3,1} := (L^{\infty} L^1)(\frac{1}{3}, 1)$. Note that the set of bounded functions $f$ such that $r^{-1} f$ belongs to $L^{3,1}$ has the same scaling as $L^\infty$.

For stating our uniqueness result, we need to introduce the set $L^\infty_L$ of locally bounded functions $f$ such that

$$\sup_{x \in \mathbb{R}^N} (\log(2 + |x|)^{-1}) |f(x)| < \infty.$$ 

Our statement reads:

**Theorem 1.2.** Let $(v^1, p^1)$ and $(v^2, p^2)$ solve the $N$-dimensional incompressible Euler equations on $[0, T]$. Assume that $v^1, v^2$ belong to $L^\infty([0, T] \times \mathbb{R}^N) \cap L^1([0, T]; C^1_\omega)$ and that $p^1, p^2$ are in $L^\infty([0, T]; L^\infty_L)$.

If, in addition, $v^1(0) = v^2(0)$ then $v^1 \equiv v^2$ on $[0, T] \times \mathbb{R}^N$.

Note that we do not need to make any decay assumption in Theorem 1.2. In particular, the assumptions are fulfilled if $v, p$ and $\omega$ are in $L^\infty([0, T] \times \mathbb{R}^3)$. Hence Theorem 1.2 is not a by-product of Vishik’s statement in [7] where some decay at infinity for the vorticity is needed.

2. The proof of global existence

The proof of Theorem 1.1 follows a standard scheme: solving (E) for regularized data, proving global a priori estimates for smooth axisymmetric solutions, then passing to the limit. Steps one and three are classical (it is only a matter of choosing a mollifier which preserves the axisymmetric structure). So we focus on the second step which relies on the following proposition:

**Proposition 2.1.** There exists a constant $C$ such that

$$\forall t \in [0, T], \quad \|\omega(t)\|_{L^\infty} \leq \|\omega^0\|_{L^\infty} \exp(C t \|r^{-1} \omega^0\|_{L^{3,1}}).$$

Proposition 2.1 is based on Lemmas 2.2 and 2.3 below.
Lemma 2.2. There exists a constant \( C \) such that \[
\left\| \frac{v_y}{r} \right\|_{L^\infty} \leq C \left\| \frac{\omega_3}{r} \right\|_{L^3,1}.
\]

Proof. According to Lemma 1 in [5], there exists a constant \( C \) such that \[
\left| v_y(x) \right| \leq C \left( \int_{|x'-x|<r} \frac{|\omega_3(x')|}{|x-x'|^2} \, dx' + r \int_{|x'-x|\geq r} \frac{|\omega_3(x')|}{|x-x'|^3} \, dx' \right).
\]

On one hand, if \( x' \in \mathbb{R}^3 \) is such that \( |x' - x| \leq r \), then \( r' \leq 2r \). On the other hand, if \( |x' - x| > r \) then we have \( r'|x-x'|^{-1} \leq 2 \). Therefore, \[
\left| v_y(x) \right| \leq 2C \int \frac{1}{|x-x'|^2} \frac{|\omega_3(x')|}{r'} \, dx'.
\]

Since \( L^{\frac{3}{2},\infty} \) is the dual set of \( L^{3,1} \) and \( y \mapsto |y|^{-2} \) belongs to \( L^{\frac{3}{2},\infty} \), we get the desired result. \( \square \)

Lemma 2.3. Let \( \alpha \) satisfy \( \partial_t \alpha + \text{div}(va) = f \) with \( v \) a smooth divergence free vector-field. Then we have \[
\forall t \in [0, T], \quad \|a(t)\|_{L^{3,1}} \leq \|a(0)\|_{L^{3,1}} + \int_0^t \|f(\tau)\|_{L^{3,1}} \, d\tau.
\]

Furthermore, equality holds if \( f \equiv 0 \).

Proof. One can assume with no loss of generality that \( f \equiv 0 \). Introducing the flow \( \psi \) of \( v \), we get \( a(t, \psi(t,x)) = a(0,x) \) for all \( (t,x) \in [0,T] \times \mathbb{R}^3 \). Therefore
\[
\left\{ y \in \Omega \mid |a(t,y)| > \lambda \right\} = \psi_t(\left\{ x \in \Omega \mid |a(0,x)| > \lambda \right\}).
\]

Due to \( \text{div} \, v = 0 \), both sets have the same measure. Now, \( L^{3,1} \) coincides with the set of functions \( g \) such that \[
\|g\|_{L^{3,1}} := \int_0^\infty \tau^{1/3} g^*(\tau) \frac{d\tau}{\tau} < \infty,
\]
where \( g^*(\tau) = \inf\{\lambda \in \mathbb{R}^+ \mid \left| \{ x \in \Omega \mid |g(x)| > \lambda \} \right| \leq \tau \} \).

This completes the proof. \( \square \)

One can now prove Proposition 2.1. As \( (\partial_t + v \cdot \nabla)(r^{-1}\omega_3) = 0 \), Lemma 2.3 gives \( \|r^{-1}\omega(t)\|_{L^{3,1}} = \|r^{-1}\omega_0\|_{L^{3,1}} \). As \( (\partial_t + v \cdot \nabla)\omega_3 = r^{-1}v_r\omega_3 \), Lemma 2.2 and Gronwall inequality yield the result.

3. The proof of uniqueness

We shall use the so-called Bony decomposition (introduced in [1]) for the product of two distributions:
\[
f g = T_f g + T_g f + R(f,g).
\]

The paraproduct operator \( T \) is defined by \( T_f g := \sum_q S_{q-1} f \Delta_q g \) and the remainder operator \( R \), by \( R(f,g) := \sum_q \Delta_q f(\Delta_{q-1} g + \Delta_{q+1} g + \Delta_{q+2} g) \). See [2] for the definition of operators \( \Delta_q \) and \( S_q \).

According to Corollary 2.5.1 in [2], that \((v,p)\) be in \( L^\infty([0,T] \times \mathbb{R}^N) \times L^\infty([0,T];L_{\infty}^{\infty}) \) guarantees that \( \nabla p = \Pi(v,w) \) where \( \Pi \) stands for the bilinear operator defined by
\[
\Pi(v,w) := -\nabla(-\Delta)^{-1} (T_{v,\omega y} \partial_j v^i + T_{v,\omega y} \partial_j w^i) + \nabla T_{i,j} R(v^i, w^j).
\]

The summation convention over repeated indices has been used, and \( T_{i,j} \) stands for the linear operator introduced by J.-Y. Chemin in [2], Theorem 2.5.1. We shall just use the fact that \( \nabla T_{i,j} \) maps \( C^{1-\varepsilon} \) in \( C^{-\varepsilon} \) for all \( \varepsilon \in (0,1) \) (see [2], Chapter 2 for the definition of Hölder spaces with negative exponents).
We claim that the bilinear operator $\Pi$ satisfies the following estimate for all $\varepsilon \in (0, 1)$:
\[
\forall q \geq -1, \quad \left\| \Delta_q \Pi(v, w) \right\|_{L^\infty} \leq C(q + 2) 2^{\|q\|_{\delta f}} \min\left( \|v\|_{C^{-\varepsilon}}, \|w\|_{C^{-\varepsilon}}, \|v\|_{C^1}, \|w\|_{C^1} \right).
\]
(3)

This may be easily deduced from (2). Indeed: we have $\Delta_q (T_{\partial_j w} \partial_j v^i) = \sum_{q_q=0}^{q+4} A_{q-q-4} \Delta_q (S_{q-1} \partial_j w^j \Delta_q \partial_j v^i)$ so that, because $\nabla (-\Delta)^{-1}$ is an homogeneous operator of degree $-1$,
\[
\left\| \Delta_q \nabla (-\Delta)^{-1} (S_{q-1} \partial_j w^j \Delta_q \partial_j v^i) \right\|_{L^\infty} \leq C 2^{-q} \left\| S_{q-1} \nabla w \right\|_{L^\infty} \left\| \Delta_q \nabla v \right\|_{L^\infty} \leq C(q + 2) 2^{q\|q\|_{\delta f}} \|w\|_{C^1} \|v\|_{C^{-\varepsilon}}.
\]

Next, standard continuity results for the paraproduct (see e.g. [2]) yield
\[
\left\| \Delta_q \nabla (-\Delta)^{-1} T_{\partial_j w} \partial_j v^i \right\|_{L^\infty} \leq C 2^{q\|q\|_{\delta f}} \|w\|_{C^1} \|v\|_{C^{-\varepsilon}}.
\]

Similar inequalities may be proved for the second term in (2). Finally, the remainder operator $R$ maps $C^{-\varepsilon} \times C^1$ in $C^{1-\varepsilon}$ provided $\varepsilon < 1$. Since $\nabla T_{l,j}$ maps $C^1$ in $C^{-\varepsilon}$, one can now conclude to inequality (3). Finally, it may be easily shown that $\Delta_q (v \cdot \nabla w)$ satisfies (3). It is only a matter of using that
\[
v \cdot w^j = T_{\partial_j w} \partial_j v^i + \partial_j (v^j, w^j) + T_{\partial_j w} w^j.
\]

We are now ready to prove Theorem 1.2. First, we notice that $\delta v := v^2 - v^1$ satisfies
\[
\partial_t \delta v + v^2 \cdot \nabla \delta v = \Pi (\delta v, v^1) + \Pi (v^2, \delta v) - \delta v \cdot \nabla v^1.
\]
(4)

With inequality (3) at our disposal, Eq. (4) may be seen as a transport equation associated to a vector-field with coefficients in $L^1([0, T]; C^1)$ and a right-hand side $\delta f$ which satisfies
\[
\| \Delta_q \delta f \|_{L^\infty} \leq C(q + 2) 2^{q\|q\|_{\delta f}} \left( \|v^1\|_{C^1} + \|v^2\|_{C^1} \right) \|\delta v\|_{C^{-\varepsilon}} \quad \text{for all } \varepsilon \in (0, 1).
\]
(5)

We notice that the function $t \mapsto V(t) := \|v^1(t)\|_{C^1} + \|v^2(t)\|_{C^1}$ belongs to $L^1([0, T])$. Therefore, by virtue of Lemma 2.5 in [3], there exists some constant $C$ such that
\[
\| \delta v(t) \|_{C^{-\varepsilon}} \leq 2 \| v^2(0) - v^1(0) \|_{C^0} \quad \text{with } \varepsilon_t := C \int_0^t V(\tau) \, d\tau \quad \text{whenever } \varepsilon_t \leq \frac{1}{2}.
\]

As $v^2(0) - v^1(0) = 0$, we get uniqueness on $[0, T_0]$ with $T_0 = \sup \{ t \in [0, T] \mid C \int_0^t V(\tau) \, d\tau \leq \frac{1}{2} \}$. Because $V \in L^1([0, T])$, the argument may be repeated so that uniqueness holds on the whole interval $[0, T]$.

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References