Abstract

Let \( k \) be a field of characteristic zero, and let \( G \) be a split simple algebraic group of type \( G_2 \) over \( k \). We prove that the functor \( R \mapsto H^1_{\text{ét}}(R, G) \) of \( G \)-torsors satisfies purity for regular local rings containing \( k \). To cite this article: V. Chernousov, I. Panin, C. R. Acad. Sci. Paris, Ser. I 345 (2007).

Résumé

Un théorème de pureté pour les \( G_2 \)-torseurs. Soit \( k \) un corps de caractéristique 0, et soit \( G \) un \( k \)-groupe simple déployé de type \( G_2 \). Nous montrons que le foncteur des \( G \)-torseurs satisfait au « théorème de pureté » pour la catégorie des anneaux locaux réguliers contenant \( k \). Pour citer cet article : V. Chernousov, I. Panin, C. R. Acad. Sci. Paris, Ser. I 345 (2007).

Version française abrégée

Soient \( \mathcal{F} \) un foncteur covariant de la catégorie des anneaux commutatifs vers celle des ensembles et \( R \) un anneau intègre de corps des fractions \( K \). On dit qu’un élément \( \xi \in \mathcal{F}(K) \) est non ramifié en l’idéal premier \( \mathfrak{p} \subset R \) de hauteur 1 si

\[
\xi \in \text{Im}[\mathcal{F}(R_{\mathfrak{p}}) \to \mathcal{F}(K)].
\]

On dit que \( \xi \) est non ramifié s’il en est non ramifié en tout idéal premier de hauteur 1 de \( R \). L’ensemble de tous les éléments non ramifiés de \( \mathcal{F}(K) \) est noté \( \mathcal{F}(K)_{\text{ur}} \). On dit que le foncteur \( \mathcal{F} \) satisfait à la condition de pureté pour un anneau intègre \( R \) si

\[
\text{Im}[\mathcal{F}(R) \to \mathcal{F}(K)] = \mathcal{F}(K)_{\text{ur}}.
\]
Dans cette note, nous considérons les deux foncteurs suivants :
\[ \mathcal{T}(R) = H^1_{et}(R, G) = \{ \text{classes d’isomorphisme des } G\text{-torseurs sur } R \} \]
où \( G \) est un groupe déployé de type \( G_2 \), et
\[ \mathcal{P}f_3(R) = \{ \text{classes d’isomorphisme de 3-formes de Pfister sur } R \} \]
Nous démontrons que ces deux foncteurs satisfont à la condition de pureté pour les anneaux locaux réguliers contenant un corps de caractéristique nulle. Cela répond affirmativement à une question posée dans [2, Question 6.4, p. 124] sur la pureté du foncteur des \( G \)-torseurs où \( G \) est un groupe déployé de type \( G_2 \).
L’hypothèse restrictive sur la caractéristique provient du fait que nous utilisons le résultat principal de [8] sur les espaces quadratiques rationnellement isotropes, qui n’a été prouvé qu’en caractéristique nulle (la preuve utilise la résolution des singularités).
Remarquons également que jusqu’à récemment, on ne trouvait aucun résultat dans la littérature sur la pureté du foncteur des \( G \)-torseurs pour les groupes de type exceptionnel. Le théorème de pureté est connu pour certains groupes de type classique : les groupes déployés de type \( A_n \) (non publié) ; les groupes de la forme \( SL_1, A \) où \( A \) est une algèbre simple centrale sur un corps [1] ; les groupes déployés de type \( B_n \) [8] ; les groupes simplement connexes déployés de type \( C_n \) (de manière évidente) ; certains groupes déployés de type \( D_n \) (comme le groupe spécial orthogonal d’une forme quadratique) [8].
La preuve de la pureté du foncteur des \( G_2 \)-torseurs consiste à montrer d’abord la pureté pour le foncteur \( \mathcal{P}f_3 \) puis que le morphisme naturel \( \mathcal{P}f_3(K)_{ur} \rightarrow \mathcal{T}(K)_{ur} \) est surjectif.

1. Main result

In this note we give an affirmative answer to a question raised in [2, Question 6.4, p. 124] about the purity of the functor of \( G \)-torsors in the case where \( G \) is a split group of type \( G_2 \).
Let us first recall what the purity property for a functor is. Let \( \mathcal{F} \) be a covariant functor from the category of commutative rings to the category of sets, and let \( R \) be a domain with field of fractions \( K \). We say that an element \( \xi \in \mathcal{F}(K) \) is unramified at a prime ideal \( \mathfrak{P} \subset R \) of height 1 if
\[ \xi \in \text{Im}\left[ \mathcal{F}(R_{\mathfrak{P}}) \rightarrow \mathcal{F}(K) \right] \]
We say that \( \xi \) is unramified if it is unramified with respect to all prime ideals in \( R \) of height 1. It is clear that
\[ \text{Im}\left[ \mathcal{F}(R) \rightarrow \mathcal{F}(K) \right] \subseteq \mathcal{F}(K)_{ur} \]
where \( \mathcal{F}(K)_{ur} \) is the set of all unramified elements. We say that the functor \( \mathcal{F} \) satisfies purity for a domain \( R \) if every \( \xi \in \mathcal{F}(K)_{ur} \) is in the image of \( \mathcal{F}(R) \), i.e. if
\[ \bigcap_{\text{ht } \mathfrak{P} = 1} \text{Im}\left[ \mathcal{F}(R_{\mathfrak{P}}) \rightarrow \mathcal{F}(K) \right] = \text{Im}\left[ \mathcal{F}(R) \rightarrow \mathcal{F}(K) \right] \]
If \( \mathcal{F} \) and \( \mathcal{F}' \) are two functors and \( f : \mathcal{F} \rightarrow \mathcal{F}' \) is a natural transformation then
\[ f(\mathcal{F}(K)_{ur}) \subseteq \mathcal{F}'(K)_{ur} \]
In what follows we assume that 2 is invertible in \( R \). We say that a quadratic space over \( R \) is a 3-Pfister space if the corresponding quadratic form is isomorphic to a form
\[ \langle a, b, c \rangle = \langle 1, -a \rangle \otimes \langle 1, -b \rangle \otimes \langle 1, -c \rangle \]
where \( a, b, c \) are units in \( R \). We will consider the following two functors:
\[ \mathcal{T}(R) = H^1_{et}(R, G) = \{ \text{isomorphism classes of } G\text{-torseurs over } R \} \]
where \( G \) is a split group of type \( G_2 \), and
\[ \mathcal{P}f_3(R) = \{ \text{isomorphism classes of 3-fold Pfister spaces over } R \} \]
The main results of this note are the following purity theorems:
**Theorem 1.** Let $G$ be a simple split algebraic group of type $G_2$ over the field $\mathbb{Q}$ of rational numbers. Then the functor $\mathcal{T} : R \mapsto H^1_{et}(R, G)$ satisfies purity for regular local rings containing $\mathbb{Q}$.

**Theorem 2.** The functor $R \mapsto \mathcal{P}_3(R)$ satisfies purity for regular local rings containing $\mathbb{Q}$.

**Remark 3.** Until recently there were no results in the literature on the purity of the functor of $G$-torsors for groups of exceptional type. For certain groups of classical type the purity theorem is known; more precisely it is known for split groups of type $A_n$ (unpublished); groups of the form $SL_{1,A}$, where $A$ is a central simple algebra over a field [1]; split groups of type $B_n$ [8]; split simply connected groups of type $C_n$ (obvious); certain split groups of type $D_n$ (like the special orthogonal group of a quadratic form) [8].

**Remark 4.** The characteristic restriction in the theorem is due to the fact that we use the main result in [8] on rationally isotropic quadratic spaces which was proven in characteristic zero only (the resolution of singularities is involved in that proof).

2. **Quadratic spaces splitting over étale quadratic extensions**

For the definition and basic properties of quadratic spaces over a commutative ring we refer to [3]. To prove purity for 3-fold Pfister forms we need information about quadratic spaces over $R$ splitting over an étale quadratic extension of $R$. In this section $R$ denotes a regular local ring. Note that $R$ is a domain; we denote by $k$ its residue field and by $K$ its field of fractions.

Let $S = R[t]/(t^2 - u)$ where $u \in R^\times$ is a unit. Let $L = S \otimes_R K$ and $l = S \otimes_R k$. It is clear that $L/K$ and $l/k$ are étale quadratic extensions of $K$ and $k$ respectively. Thus $L/K$ is either isomorphic to $K \times K$ or it is a Galois field extension of degree 2. Let $\sigma \in Gal(S/R)$ be the nontrivial automorphism. It induces the involutions of $L/K$ and $l/k$ which we denote for simplicity by the same letter $\sigma$.

If $(V, q)$ is a quadratic space over $R$ we set

$$V_S = (V_S, q_S) = (V, q) \otimes_R S, \quad V_L = (V_L, q_L) = (V_S, q_S) \otimes_S L, \quad V_K = (V_K, q_K) = (V, q) \otimes_R K, \quad V_k = (V_k, q_k) = (V, q) \otimes_R k, \quad V_l = (V_l, q_l) = (V_S, q_S) \otimes_R k.$$

If $w \in V_S$ and $v \in V$ we set

$$\bar{w} = w \otimes 1 \in V_l, \quad \bar{v} = v \otimes 1 \in V_k.$$

**Lemma 5.** Let $(W, q)$ be a quadratic space over $R$ of rank 2 such that $W_S$ is a hyperbolic plane $\mathbb{H}_S$. Then either $W$ is a hyperbolic plane or there exists a unit $\lambda \in R^\times$ such that $q \cong \lambda \cdot (1, -u)$.

**Proof.** Since $R$ is local, we can write $q$ in a diagonal form $q = \lambda \cdot (1, -d)$ where $\lambda, d \in R^\times$. Assume first that $q_K$ is hyperbolic. Then there is $f \in K$ such that $f^2 = d$. Since $R$ is a unique factorization domain ([4, Theorem 48, page 142]) and $d$ is a unit, we have $f \in R^\times$. Thus $q$ splits over $R$.

Let now $q_K$ be anisotropic over $K$. Note that $L/K$ is a field extension since otherwise $L = K \times K$ and hence $q_L = q_K \times q_K$ would be $L$-anisotropic. Since $q_K$ splits over $L = K(\sqrt{u})$ the discriminant of $q_K$ is $u(K^\times)^2$. It follows that $d f^2 = u$ for some $f \in K^\times$ or equivalently $f^2 = d^{-1} u \in R^\times$. As above, this implies $f \in R^\times$, hence $q \cong \lambda \cdot (1, -u)$.

**Proposition 6.** Let $(V, q)$ be a quadratic space over $R$. Assume that the space $(V_S, q_S)$ is hyperbolic. Then $(V, q)$ can be decomposed as

$$(V, q) = (W_1, q_1) \perp \cdots \perp (W_n, q_n)$$

where $(W_i, q_i), i = 1, \ldots, n$, are quadratic spaces of rank 2 splitting over $S$. 

Proof. We argue by induction on $n$ where $\dim V = 2n$. If $\dim V = 2$ there is nothing to prove. Assume $\dim V \geq 3$. Take a unimodular vector $v \in V_S$ with $q_S(v) = 0$. Since $V_l$ is a hyperbolic space of rank at least 3 there is a vector $\bar{w} \in V_l$ such that

$$q_l(\bar{w}) = 0, \quad (\bar{w}, \sigma(\bar{w})) \in I^\times, \quad (\bar{w}, \bar{u}) \in I^\times$$

where $(-, -)$ is the bilinear form associated with $q$. It is clear that $\bar{w}$ and $\sigma(\bar{w})$ generate a hyperbolic plane over $l$ which is a direct summand of $V_l$.

Take now any lifting $w$ of $\bar{w}$. Replacing $w$ by

$$w - \frac{(w, w)}{2(w, v)} v$$

we may additionally assume that $q_S(w) = 0$. Set $W = wS + \sigma(w)S \subseteq V_S$. Since $(w, \sigma(w)) \in S^\times$, the space $W$ is a $\sigma$-invariant hyperbolic plane, and it is clear that $(W, q_S|_W)$ is a direct summand of $(V_S, q_S)$.

Thus we have orthogonal quadratic space decompositions $V_S = W \oplus W^\perp$ and $V = W^\sigma \oplus (W^\perp)^\sigma$, where $W^\perp$ is the orthogonal complement of $W$ and $W^\sigma$, $(W^\perp)^\sigma$ are $\sigma$-invariant subspaces. We also have $W^\sigma \otimes_R S = W$ and $(W^\perp)^\sigma \otimes_R S = W^\perp$ since $V \otimes_R S = V_S$. The first equality implies that $W^\sigma$ splits over $S$. Since $S$ is semi-local and $V_S$ is hyperbolic, so is $W^\perp$. Applying the induction hypothesis to $(W^\perp)^\sigma$ completes the proof. \( \square \)

Proposition 6 and Lemma 5 imply the following:

**Corollary 7.** Let $(V, q)$ be a quadratic space over $R$ such that $(V_S, q_S)$ is hyperbolic. If $(V_K, q_K)$ is anisotropic, then there exist units $\lambda_1, \lambda_2, \ldots, \lambda_n \in R^\times$ and a quadratic space isomorphism $q \cong \langle \lambda_1, \lambda_2, \ldots, \lambda_n \rangle \otimes (1, -u)$.

3. Purity of 3-fold Pfister forms

In this section we keep the above notation. In addition, we assume $\mathbb{Q} \subset R$.

**Theorem 8.** Let $\phi$ be a 3-fold Pfister form over $K$. Assume that $\phi$ is unramified over $K$ viewed as an 8-dimensional quadratic form. Then there exists a Pfister space $\phi' := (\langle a', b', c' \rangle)$ over $R$, where $a', b', c' \in R^\times$, such that $\phi' \otimes_R K \cong \phi$.

Proof. Reasoning first as in [6, Proof of Theorem A] and then as in [6, Proof of Theorem 7.1] we may reduce the general case to the case of a regular local ring $R$ of the form $O_{X,x}$, where $X$ is a smooth affine variety over a field $k$ of characteristic zero and $x \in X$ is a closed point.

If $\phi$ is isotropic, then it is hyperbolic and there is nothing to prove. So we may assume that $\phi$ is anisotropic.

By [8, Cor. 1.0.5], there exist a quadratic space $\psi$ over $R$ such that $\psi_K$ is isomorphic to $\phi$. Since $\psi_K$ represents 1 over $K$ there is a decomposition $\psi \cong (1) \perp \psi'$ over $R$, by [8, Cor. 1.0.6]. Take a diagonalization $\psi' = \langle u_2, u_3, \ldots, u_8 \rangle$ of $\psi'$ over $R$ and set $u := -u_2$.

Let $S$ and $L$ be the étale quadratic extensions of $R$ and $K$ respectively corresponding to $u$. Since $\psi$ is isotropic over $S$, hence over $L$, so is $\psi$. It then follows that $\psi_K = \phi$ is hyperbolic over $L$. We claim that the space $\psi \otimes_R S$ is hyperbolic. Indeed, the ring $S$ is regular semi-local of the form $O_{Y,y}$, where $Y$ is a $k$-smooth affine variety and $y \subset Y$ is a subset consisting of one or two closed points. If $S$ is local, i.e. $y$ is a closed point, the hyperbolicity of $\psi \otimes_R S$ was proven by Ojanguren [5]. This result was reproved by Ojanguren, Panin in [7, Proof of Theorem 5.1] and we note that the arguments therein work equally well for semi-local regular rings of a finite family of closed points on a $k$-smooth variety.

Since $\psi_K$ is anisotropic, by Corollary 7 we can write $\psi$ in the form

$$\psi = \langle 1, -u \rangle \otimes \langle b_1, b_2, b_3, b_4 \rangle = b_1 \psi_1 \perp b_3 \psi_2$$

where $b_1, b_2, b_3, b_4 \in R^\times$ and

$$\psi_1 = \langle 1, -u \rangle \otimes (1, b_1 b_2), \quad \psi_2 = \langle 1, -u \rangle \otimes (1, b_2 b_4).$$

We claim that $\psi_1 \cong \psi_2$. Indeed, by a theorem of Ojanguren [5] it suffices to check this over $K$. The function field $K(\psi_1)$ of the quadric $\psi_{1,K} = 0$ splits both $\psi_{1,K}$ and $\psi_K$. By Witt cancellation it splits $\psi_{2,K}$ as well. It follows that
ψ₁ and ψ₂ are K-isomorphic, since they are 2-fold Pfister forms. Thus ψ can be written in the form ψ = b₁η where η is the 3-fold Pfister form

η = (1, b₁b₂) ⊗ ψ₁.

It remains to show that b₁η is isomorphic to η over R or equivalently, by a theorem of Ojanguren [5], over K. In turn, this is equivalent to the property that η_K represents b₁. But this is clear, since ψₖ = b₁η and ψ represents 1.

Corollary 9. The functor Pf₃ satisfies purity for regular local rings containing Q.

4. Proof of Theorem 1

It is known that a split R-group G of type G₂ is the automorphism group of a split octonion algebra O and this gives rise to an embedding i : G \hookrightarrow O₈ of G into a split orthogonal group O₈ in dimension 8. Let 1, e₁, e₂, ..., e₇ be a canonical basis of O with

\[ e₁^2 = 1, \quad e₂^2 = 1, \quad e₃^2 = 1, \]
\[ e₄ = e₁e₂ = -e₂e₁, \quad e₅ = e₂e₃ = -e₃e₂, \]
\[ e₆ = e₃e₄ = -e₄e₃, \quad e₇ = e₄e₅ = -e₅e₄. \]

It follows from these relations that a mapping

\[ e₁ \mapsto e₁e₁, \quad e₂ \mapsto e₂e₂, \quad e₃ \mapsto e₃e₃ \]

where e₁, e₂, e₃ ∈ {±1} can be extended to an automorphism of O and this gives rise to an embedding j : μ₂ × μ₂ × μ₂ \hookrightarrow G.

Consider the following diagram of pointed sets

\[ H^1_{\text{ét}}(K, \mu_2 \times \mu_2 \times \mu_2) \xrightarrow{j_K} H^1_{\text{ét}}(K, G) \xrightarrow{i_K} H^1_{\text{ét}}(K, O₈) \]
\[ \alpha \quad \beta \quad \gamma \]
\[ H^1_{\text{ét}}(R, \mu_2 \times \mu_2 \times \mu_2) \xrightarrow{j_R} H^1_{\text{ét}}(R, G) \xrightarrow{i_R} H^1_{\text{ét}}(R, O₈). \]

Recall that the set H^1_{\text{ét}}(R, O₈) (resp. H^1_{\text{ét}}(K, O₈)) classifies isomorphism classes of quadratic forms of dimension 8 over R (resp. over K). It is known [9, Th. 1.7.1, page 16] that i_K is injective and the image of i_K coincides with the set of isomorphism classes of 3-fold Pfister forms. Since R is local, we have H^1_{\text{ét}}(R, μ₂) ≃ R^×/R^×2. If a, b, c are units in R (resp. in K), we write (a, b, c) for the corresponding element in H^1_{\text{ét}}(R, μ₂ × μ₂ × μ₂) (resp. in H^1_{\text{ét}}(K, μ₂ × μ₂ × μ₂)). It follows from our construction of j that (i_R ∘ j_R)(a, b, c) (resp. (i_K ∘ j_K)(a, b, c)) is the Pfister space \langle\langle a, b, c\rangle\rangle over R (resp. K).

Let \xi ∈ H^1_{\text{ét}}(K, G) be an unramified class, and let φ be a quadratic form over K representing the class i_K(ξ). We mentioned above that φ is a 3-fold Pfister form. Since

\[ i_K(H^1(K, G)_{\text{ur}}) ⊂ H^1(K, O₈)_{\text{ur}}, \]

the class [φ] is unramified. By Theorem 8 we have a K-quadratic space isomorphism φ ∼ \langle\langle a, b, c\rangle\rangle for some a, b, c ∈ R^×. Set \̂ξ = j_R(a, b, c). We claim that \̂ξ_K = β(ξ) = ξ. Indeed,

\[ i_K(\̂ξ_K) = (i_R(j_R(a, b, c))) = (γ ∘ i_R ∘ j_R)(a, b, c) = φ = i_K(ξ). \]

Since i_K is injective, \̂ξ_K = ξ. This completes the proof.

Remark 10. J.-P. Serre pointed out to us that if R is any local ring in which 2 is invertible then the following three categories are equivalent:

(i) the category of G₂-torsors over R;
(ii) the category of isomorphism classes of octonion algebras over R;
(iii) the category of isomorphism classes of 3-fold Pfister spaces over R.
This shows that Theorem 1 is a straightforward consequence of Corollary 9.

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References