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Mathematical Problems in Mechanics/Numerical Analysis

Sharp error estimates for a fractional-step method applied to the 3D Navier–Stokes equations ☆

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Abstract

We present some improvements on the error estimates obtained by J. Blasco and R. Codina for a viscosity-splitting in time scheme, with finite element approximation, applied to the Navier–Stokes equations. The key is to obtain new error estimates for the discrete in time derivative of velocity, which let us reach, in particular, an error of order one (in time and space) for the pressure approximation. ***To cite this article: F. Guillén-González, M.V. Redondo-Neble, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

Estimations optimales d'erreur d'une méthode de pas fractionnaire appliqué à des équations de Navier–Stokes 3D. On présente quelques améliorations sur les estimations d'erreur obtenues par J. Blasco et R. Codina pour une discrétisation des équations de Navier–Stokes par un schéma de décomposition en temps et des éléments finis en espace. L'idée principale est l'obtention de nouvelles estimations d'erreur de la dérivée discrète en temps de la vitesse, qui entraînent des estimations d'erreur d'ordre un (en temps et en espace) pour l'approximation de la pression. ***Pour citer cet article : F. Guillén-González, M.V. Redondo-Neble, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

1. Introduction

We consider the unsteady, incompressible Navier–Stokes equations in a bounded domain $\Omega \subset I\mathbb{R}^3$:

$$(P)\quad \begin{cases} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f}, \quad \nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega\times(0,T), \\ \mathbf{u} = 0 \quad \text{on } \partial\Omega\times(0,T), \quad \mathbf{u}_{|t=0} = \mathbf{u}_0 \quad \text{in } \Omega, \end{cases}$$

where $\mathbf{u}(\mathbf{x},t)$ is the fluid velocity at position $\mathbf{x}\in\Omega$ and time $t\in(0,T)$, $p(\mathbf{x},t)$ the pressure, $\nu>0$ the viscosity (which is assumed constant) and $\mathbf{f}$ the external force.

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Fractional step methods are widely used to approximate the problem (P). They allow us to separate the effects of different operators appearing in the problem. The origin of these methods is the well known Chorin–Temam projection scheme, where the idea is to avoid the computation of the Stokes problem with a two step scheme: first step is a Dirichlet-elliptic problem for an intermediate velocity and the second one is a free divergence projection step equivalent to a Neumann-elliptic problem for the pressure.

Error estimates for projection methods can be seen in [6,7] for time discrete schemes and in [4] for a fully discrete scheme.

In this Note, we will study a viscosity-splitting scheme, introduced and studied by J. Blasco and R. Codina [1–3]. It is a two-step scheme, where the main numerical difficulties of (P) (namely, the treatment of nonlinear term $(\mathbf{u}\cdot\nabla)\mathbf{u}$ and the relation between incompressibility $\nabla\cdot\mathbf{u}=0$ and pressure), are split into two different steps, considering the diffusive terms in both steps.

Notice that both type of schemes, projection and viscosity-splitting schemes, can be jointly used, because the second step of the viscosity-splitting scheme could be computed with a projection method.

We use a finite element approximation. Let Ω be a 3D polyhedron (or a 2D polygon) such that Stokes problem in Ω has $\mathbf{H}^2\times H^1$ regularity for velocity and pressure respectively.

We consider a partition $\{t_m=m\,k\}$ with $k=T/M$ of the time interval $[0,T]$, and a family of finite dimensional spaces $\mathbf{V}_h\subset\mathbf{H}_0^1(\Omega)$ and $Q_h\subset L_0^2(\Omega)$ defined from finite element methods of the domain Ω of mesh size h. $\mathbf{V}_h$ and Q_h are thus required to satisfy:

- the Brezzi–Babuska stability condition: $\inf_{q_h\in Q_h\setminus\{0\}}(\sup_{\mathbf{v}\in\mathbf{V}_h\setminus\{0\}}\frac{(q_h,\nabla\cdot\mathbf{v}_h)}{\|\mathbf{v}_h\|\,|q_h|})\geqslant\beta$;
- the approximating properties:

$$\frac{1}{h}\inf_{\mathbf{v}_h\in\mathbf{V}_h}|\mathbf{u}-\mathbf{v}_h|+\inf_{\mathbf{v}_h\in\mathbf{V}_h}\|\mathbf{u}-\mathbf{v}_h\|+\inf_{q_h\in Q_h}|p-p_h|\leqslant Ch\big\|(\mathbf{u},p)\big\|_{H^2\times H^1}.$$

We use the notation $|\cdot|$ and $(\cdot,\cdot)$ as the norm and the inner product in L^2 and $\|\cdot\|$ as the norm in H_0^1. The fully discrete scheme is described as follows:
Initialization: Let $\mathbf{u}_h^0\in\mathbf{V}_h$ be an approximation of $\mathbf{u}_0$
Step of time $m+1$:

Substep 1: Given $\mathbf{u}_h^m\in\mathbf{V}_h$, to compute $\mathbf{u}_h^{m+1/2}\in\mathbf{V}_h$ such that,

$$(S_1)_h^{m+1}\quad \frac{1}{k}\big(\mathbf{u}_h^{m+1/2}-\mathbf{u}_h^m,\mathbf{v}_h\big)+c\big(\mathbf{u}_h^m,\mathbf{u}_h^{m+1/2},\mathbf{v}_h\big)+\big(\nabla\mathbf{u}_h^{m+1/2},\nabla\mathbf{v}_h\big)=\big(\mathbf{f}^{m+1},\mathbf{v}_h\big),\quad\forall\mathbf{v}_h\in\mathbf{V}_h$$

where $c(\mathbf{w},\mathbf{u},\mathbf{v})=\{(\mathbf{w}\cdot\nabla\mathbf{u},\mathbf{v})-(\mathbf{w}\cdot\nabla\mathbf{v},\mathbf{u})\}/2$.
Substep 2: Given $\mathbf{u}_h^{m+1/2}$, to compute $(\mathbf{u}_h^{m+1},p_h^{m+1})\in\mathbf{V}_h\times Q_h$ such that,

$$(S_2)_h^{m+1}\quad\begin{cases}\frac{1}{k}\big(\mathbf{u}_h^{m+1}-\mathbf{u}_h^{m+1/2}\mathbf{v}_h\big)+\big(\nabla\big(\mathbf{u}_h^{m+1}-\mathbf{u}_h^{m+1/2}\big),\nabla\mathbf{v}_h\big)-\big(p_h^{m+1},\nabla\cdot\mathbf{v}_h\big)=0,\quad\forall\mathbf{v}_h\in\mathbf{V}_h,\\ \big(\nabla\cdot\mathbf{u}_h^{m+1},q_h\big)=0,\quad\forall q_h\in Q_h.\end{cases}$$

With respect to the effective computation of this scheme, in each time step, it will be necessary to compute (i) $(S_1)_h^{m+1}$ as three discrete linear convection–diffusion equations (the system is uncoupled). (ii) $(S_2)_h^{m+1}$ as a discrete Stokes problem.

Assuming the following regularity for the exact solution $(\mathbf{u},p)$

$$(R1)\quad \mathbf{u}\in L^\infty\big(\mathbf{H}^2\cap\mathbf{V}\big),\quad p\in L^\infty\big(H^1\big),\quad \mathbf{u}_t\in L^\infty\big(\mathbf{L}^2\big)\cap L^2\big(\mathbf{H}^1\big),\quad \mathbf{u}_{tt}\in L^2(\mathbf{V}')$$

(here $\mathbf{V}=\{\mathbf{v}\in\mathbf{H}_0^1\colon\ \nabla\cdot\mathbf{v}=0\text{ in }\Omega,\ \mathbf{v}=0\text{ on }\partial\Omega\}$), and the constraints $h^2\leqslant Ck$, the following error estimates hold [1,3], $\|\mathbf{u}(t_m)-\mathbf{u}_h^m\|_{l^\infty(L^2)\cap l^2(H^1)}\leqslant C(k+h)$ and $\|p(t_m)-p^m\|_{l^2(L^2)}\leqslant C\sqrt{k}$ (here, p^m is the pressure of the corresponding time discrete scheme). Moreover, the estimate $\|p^m-p_h^m\|_{l^2(L^2)}\leqslant Ch/\sqrt{k}$ can be obtained with similar arguments.

The objectives of this work are:

1. To improve the order of error estimate in pressure, from $O(\sqrt{k}+h/\sqrt{k})$ to $O(k+h)$.
2. To improve the norm of error estimates in velocity and pressure, concretely from $l^\infty(L^2)$ to $l^\infty(H^1)$ in velocity and from $l^2(L^2)$ to $l^\infty(L^2)$ in pressure.
3. To improve the order of error estimate in velocity in norm $L^2(\mathbf{L}^2)$, from $O(k+h)$ to $O(k+h^2)$.

The main result of this paper can be written as follows. Assuming additional regularity hypotheses:

$$(R2)\quad p_t\in L^2(H^1),\ \mathbf{u}_t\in L^\infty(\mathbf{H}^1)\cap L^2(\mathbf{H}^2),\ \mathbf{u}_{tt}\in L^2(\mathbf{L}^2)\cap L^\infty(\mathbf{H}^{-1}),\ \mathbf{u}_{ttt}\in L^2(\mathbf{V}'),\ \sqrt{t}\mathbf{u}_{ttt}\in L^2(\mathbf{H}^{-1})$$

then, $\|p(t_m)-p_h^m\|_{l^2(L^2)}\leqslant C(k+h)$, $\|\mathbf{u}(t_m)-\mathbf{u}_h^m\|_{l^2(\mathbf{L}^2)}\leqslant C(k+h^2)$. Moreover, assuming the following hypothesis for initial step:

$$\left|(\mathbf{u}(t_1)-\mathbf{u}^1)-(\mathbf{u}(t_0)-\mathbf{u}^0)\right|\leqslant Ck^2,\qquad \left|(\mathbf{u}^1-\mathbf{u}_h^1)-(\mathbf{u}^0-\mathbf{u}_h^0)\right|\leqslant Ckh$$

then

$$\left\|\mathbf{u}(t_m)-\mathbf{u}_h^m\right\|_{l^\infty(\mathbf{H}^1)}\leqslant C(k+h),\qquad \left\|p(t_m)-p_h^m\right\|_{l^\infty(L^2)}\leqslant C(k+h).$$

Therefore, we have that this scheme has the same analytical results than Euler's type schemes, improving their numerical treatment (since the main difficulties are split). In the following two sections, we will present an outline of the proof (see [5] for a complete explanation of the results).

Unfortunately, in order to assure the additional regularity hypotheses $(R2)$, it is necessary to assume that $\mathbf{u}_t(0)\in\mathbf{H}^1$, which implies a nonlocal compatibility condition for the data $\mathbf{u}_0$ and $\mathbf{f}$ [8]. In this sense, we could relax it approximating the first step with several auxiliary initial steps with a sufficiently small time step. Then, the approximate solutions obtained from these preliminary steps could serve as initial data for our fractional step algorithm at subsequent time steps.

2. Error estimates for the time discrete scheme

We decompose the total error in their temporal and spatial parts, introducing the corresponding time discrete scheme (where the discrete spaces $(\mathbf{V}_h,Q_h)$ must be changed by $(\mathbf{H}_0^1,L_0^2)$), which problems are denoted as $(S_1)^{m+1}$ and $(S_2)^{m+1}$ and the solutions are denoted as $\mathbf{u}^{m+1/2}$ and $(\mathbf{u}^{m+1},p^{m+1})$ respectively. We introduce the following notations for the time discrete errors in $t=t_{m+1}$:

$$\mathbf{e}^{m+1/2}=\mathbf{u}(t_{m+1})-\mathbf{u}^{m+1/2},\quad \mathbf{e}^{m+1}=\mathbf{u}(t_{m+1})-\mathbf{u}^{m+1},\quad e_p^{m+1}=p(t_{m+1})-p^{m+1},$$

and for the discrete in time derivatives of errors

$$\delta_t\mathbf{e}^{m+1}=\frac{\mathbf{e}^{m+1}-\mathbf{e}^m}{k},\qquad \delta_t\mathbf{e}^{m+1/2}=\frac{\mathbf{e}^{m+1/2}-\mathbf{e}^{m-1/2}}{k}.$$

Finally, the problems verified by the errors $\mathbf{e}^{m+1/2}$ and $(\mathbf{e}^{m+1},e_p^{m+1})$ will be denoted by $(E_1)^{m+1}$ and $(E_2)^{m+1}$ respectively, and $(E_3)^{m+1}$ obtained by adding $(E_1)^{m+1}$ and $(E_2)^{m+1}$ (see the next section for the problems verified by the space discrete errors).

Theorem 2.1. *The following error estimate holds* (*for k small enough*): $\|e_p^{m+1}\|_{l^2(L^2)}\leqslant Ck$.

Proof. It is based on the following three steps:

(i) H^2 error estimates. Using the $H^2\times H^1$-regularity of Stokes problem verified by $(\mathbf{e}^{m+1},e_p^{m+1})$ and the H^2-regularity of the Poisson–Dirichlet problem verified by $\mathbf{e}^{m+1/2}$, one has $\mathbf{e}^{m+1/2},\mathbf{e}^{m+1}$ are bounded in $l^\infty(\mathbf{H}^2)$.

(ii) Making $(\delta_t(E_1)^{m+1},\delta_t\mathbf{e}^{m+1/2})+(\delta_t(E_2)^{m+1},\delta_t\mathbf{e}^{m+1})$, one gets

$$\left\|\delta_t\mathbf{e}^{m+1}\right\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)}+\left\|\delta_t\mathbf{e}^{m+1/2}\right\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)}\leqslant Ck^{1/2}.$$

(iii) Duality argument. Making $(\delta_t(E_3)^{m+1},A^{-1}\delta_t\mathbf{e}^{m+1})$, A being the Stokes operator, one has

$$\left\|\delta_t\mathbf{e}^{m+1}\right\|_{l^\infty(\mathbf{V}')\cap l^2(\mathbf{L}^2)}\leqslant Ck$$

for k small enough (which becomes from to apply the generalized discrete Gronwall's lemma). □

Theorem 2.2. *Assuming $|\delta_t \mathbf{e}^1| \leqslant C\,k$, the following error estimates hold $\|\mathbf{e}^{m+1}\|_{l^\infty(\mathbf{H}^1)} \leqslant Ck$ and $\|e_p^{m+1}\|_{l^\infty(L^2)} \leqslant Ck$.*

Proof. It is based on the error estimate $\|\delta_t \mathbf{e}^{m+1}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)} \leqslant C\,k$, obtained by making $(\delta_t(S_3)^{m+1}, \delta_t \mathbf{e}^{m+1})$. □

3. Error estimates for the spatial discretization

We define the space discrete errors as: $\mathbf{e}_d^{m+1} = \mathbf{u}^{m+1} - \mathbf{u}_h^{m+1}$, $\mathbf{e}_d^{m+1/2} = \mathbf{u}^{m+1/2} - \mathbf{u}_h^{m+1/2}$, $e_{p,d}^{m+1} = p^{m+1} - p_h^{m+1}$. Then, the problems verified by the space discrete errors are:

$$(E_1)_h^{m+1} \quad \frac{1}{k}\big(\mathbf{e}_d^{m+1/2} - \mathbf{e}_d^m, \mathbf{v}_h\big) + \big(\nabla \mathbf{e}_d^{m+1/2}, \nabla \mathbf{v}_h\big) = \mathbf{NL}_h^{m+1}(\mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{V}_h,$$

where $\mathbf{NL}_h^{m+1}(\mathbf{v}_h) = c(\mathbf{e}_d^m, \mathbf{u}^{m+1/2}, \mathbf{v}_h) - c(\mathbf{u}_h^m, \mathbf{e}_d^{m+1/2}, \mathbf{v}_h)$, and

$$(E_2)_h^{m+1} \quad \begin{cases} \frac{1}{k}\big(\mathbf{e}_d^{m+1} - \mathbf{e}_d^{m+1/2}, \mathbf{v}_h\big) + \big(\nabla\big(\mathbf{e}_d^{m+1} - \mathbf{e}_d^{m+1/2}\big), \nabla \mathbf{v}_h\big) - \big(e_{p,d}^{m+1}, \nabla \cdot \mathbf{v}_h\big) = 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \\ \big(\nabla \cdot \mathbf{e}_d^{m+1}, q_h\big) = 0, \quad \forall q_h \in Q_h. \end{cases}$$

Adding $(E_1)_h^{m+1}$ and $(E_2)_h^{m+1}$, one has:

$$(E_3)_h^{m+1} \quad \big(\delta_t \mathbf{e}_d^{m+1}, \mathbf{v}_h\big) + \big(\nabla \mathbf{e}_d^{m+1}, \nabla \mathbf{v}_h\big) - \big(e_{p,d}^{m+1}, \nabla \cdot \mathbf{v}_h\big) = \mathbf{NL}_h^{m+1}(\mathbf{v}_h), \quad \big(\nabla \cdot \mathbf{e}_d^{m+1}, q_h\big) = 0.$$

Theorem 3.1. *For k small enough, the following error estimate holds*: $\|\mathbf{e}_d^{m+1}\|_{l^2(\mathbf{L}^2)} \leqslant C(k + h^2)$

Proof. It is based on the following steps: (i) Making $(\delta_t(E_1)^{m+1}, \delta_t \mathbf{e}_h^{m+1/2}) + (\delta_t(E_2)^{m+1}, \delta_t \mathbf{e}_h^{m+1})$, one can arrives at $\|\delta_t \mathbf{e}_d^{m+1}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)} + \|\delta_t \mathbf{e}_d^{m+1/2}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)} \leqslant C$. (ii) Additional scheme estimates: $\mathbf{u}_h^{m+1}$ is bounded in $l^\infty(\mathbf{W}^{1,3} \cap \mathbf{L}^\infty)$. (iii) Duality argument. Making $((E_3)_h^{m+1}, A_h^{-1}\,\mathbf{e}_h^{m+1})$, A_h being the discrete Stokes operator. □

Theorem 3.2. *For k small enough, $\|e_{p,d}^{m+1}\|_{l^2(L^2)} \leqslant Ch$.*

Proof. It is based on the following estimate $\|\delta_t \mathbf{e}_d^{m+1}\|_{l^2(\mathbf{L}^2)} \leqslant Ch$, obtained by making $(\delta_t(E_3)_h^{m+1}, A_h^{-1}\,\delta_t \mathbf{e}_h^{m+1})$. □

Theorem 3.3. *Assuming $|\delta_t \mathbf{e}_d^1| \leqslant Ch$, then $\|\mathbf{e}_d^{m+1}\|_{l^\infty(\mathbf{H}^1)} \leqslant Ch$ and $\|e_{p,d}^{m+1}\|_{l^\infty(L^2)} \leqslant Ch$.*

Proof. It is based on the error estimate $\|\delta_t \mathbf{e}_d^{m+1}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)} + \|\delta_t \mathbf{e}_d^{m+1/2}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)} \leqslant Ch$, obtained as in step (i) of the proof of Theorem 3.1. □

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