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Algebraic Geometry

Fourier–Mukai transforms of curves and principal polarizations

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Abstract

Given a Fourier–Mukai transform $\Phi : \mathbf{D}(C) \to \mathbf{D}(C')$ between the bounded derived categories of two smooth projective curves, we verify that the induced map $\phi : J(C) \to J(C')$ between the Jacobian varieties preserves the principal polarization if and only if Φ is an equivalence. ***To cite this article: M. Bernardara, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

Transformées de Fourier–Mukai et polarisations principales. Soit $\Phi : \mathbf{D}(C) \to \mathbf{D}(C')$ une transformation de Fourier–Mukai entre les catégories dérivées bornées de deux courbes lisses projectives. On vérifie que l'application $\phi : J(C) \to J(C')$ induite entre les variétés jacobiennes préserve les polarisations principales si et seulement si Φ est une équivalence. ***Pour citer cet article : M. Bernardara, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Version française abrégée

Soient C et C' deux courbes projectives lisses complexes. On se donne un foncteur de Fourier–Mukai $\Phi_{\mathcal{E}} : \mathbf{D}(C) \to \mathbf{D}(C')$, de noyau $\mathcal{E}$, entre les catégories dérivées bornées des faisceaux cohérents des deux courbes. Le foncteur $\Phi_{\mathcal{E}}$ est défini de la façon suivante

$$\begin{aligned} \Phi_{\mathcal{E}} : \mathbf{D}(C) &\longrightarrow \mathbf{D}(C'), \\ \mathcal{F} &\longmapsto Rq_*(p^*\mathcal{F} \otimes \mathcal{E}), \end{aligned}$$

où p et q sont les projections de $C \times C'$ sur C et C' respectivement et $\mathcal{E}$ est un objet de la catégorie dérivée bornée des faisceaux cohérents $\mathbf{D}(C \times C')$ sur le produit des deux courbes.

On veut décrire un morphisme $\phi_e^J : J(C) \to J(C')$ entre les variétés jacobiennes des deux courbes qui soit induit par le foncteur $\Phi_{\mathcal{E}}$. Pour cela, on décrit l'appplication affine $\Phi_e^P : \mathrm{Pic}_{\mathbb{Q}}(C) \to \mathrm{Pic}_{\mathbb{Q}}(C')$ entre les groupes de Picard à coefficients rationnels des deux courbes induite par le foncteur. La linéarisation de cette application affine, restreinte au degré zéro, donne un morphisme $\phi_e^{J_{\mathbb{Q}}} : J_{\mathbb{Q}}(C) \to J_{\mathbb{Q}}(C')$. Il y a donc un seul morphisme de variétés abéliennes $\phi_e^J : J(C) \to J(C')$ qui correspond à $\phi_e^{J_{\mathbb{Q}}}$ quand on se restreint aux coefficients rationnels.

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Plus en détail, étant donné un élément $\mathcal{E}$ de $\mathbf{D}(C \times C')$ et sa classe e dans le groupe de Grothendieck $K(C \times C')$, on définit

$$\begin{aligned} \Phi_e^P : \mathrm{Pic}_{\mathbb{Q}}(C) &\longrightarrow \mathrm{Pic}_{\mathbb{Q}}(C'), \\ M &\longmapsto q_*\left(p^*M.c_1(e) - \frac{1}{2}c_1(e).p^*K_C + \frac{1}{2}\big(c_1^2(e) + 2c_2(e)\big)\right). \end{aligned}$$

L'application Φ_e^P est celle induite par le foncteur $\Phi_{\mathcal{E}}$. On le montre en utilisant le Théorème de Grothendieck–Riemann–Roch.

On définit ϕ_e^J de la façon suivante

$$\begin{aligned} \phi_e^J : J(C) &\longrightarrow J(C') \\ M &\longmapsto q_*\big(p^*(M - \mathcal{O}_C).c_1(e)\big). \end{aligned}$$

Si on linéarise Φ_e^P et on se restreint au groupe $\mathrm{Pic}^0_{\mathbb{Q}}(C)$, on obtient le même morphisme qu'on obtient en restreignant ϕ_e^J. On peut vérifier que ϕ_e^J est le seul morphisme entre les variétés abéliennes qui soit compatible avec Φ_e^P, et donc avec $\Phi_{\mathcal{E}}$, et que la correspondance entre ϕ_e^J et $\Phi_{\mathcal{E}}$ est bien fonctorielle.

Considérons maintenant les variétés jacobiennes $J(C)$ et $J(C')$ et leurs polarisations principales $\theta_C : J(C) \to \hat{J}(C)$ et $\theta_{C'} : J(C') \to \hat{J}(C')$. Considérons aussi le morphisme ϕ_e^J et son dual $\hat{\phi}_e^J$. Pour vérifier que ϕ_e^J préserve les polarisations principales, il suffit de vérifier que $\hat{\phi}_e^J \circ \phi_e^J$ est l'identité sur $J(C)$, modulo l'identification de $J(C)$ et $J(C')$ avec leurs duales respectives via les polarisations respectives.

Il s'avère que le foncteur $\Phi_{\mathcal{E}_L}$, adjoint à gauche de $\Phi_{\mathcal{E}}$, induit le morphisme $\hat{\phi}_e^J$. Comme l'adjoint d'une équivalence est sa quasi-inverse, on peut donc conclure que le morphisme ϕ_e^J preserve les polarisations principales si et seulement si le foncteur $\Phi_{\mathcal{E}}$ est une équivalence.

Rappelons finalement que quand on a une équivalence $\Phi_{\mathcal{E}} : \mathbf{D}(C) \to \mathbf{D}(C')$ entre les catégories dérivées de deux courbes lisses projectives complexes, on en déduit toujours un isomorphisme $f : C \to C'$. De même le Théorème de Torelli entraîne que quand on a un isomorphisme de variétés abéliennes principalement polarisées $\phi : J(C) \to J(C')$ entre les jacobiennes de deux telles courbes, on en déduit toujours un isomorphisme $f : C \to C'$. On a donc établi une correspondance entre le Théorème de Torelli et la caracterisation d'une courbe lisse par sa catégorie dérivée.

1. Introduction

Let C and C' be two smooth complex projective curves. Consider a Fourier–Mukai functor $\Phi_{\mathcal{E}} : \mathbf{D}(C) \to \mathbf{D}(C')$ between the bounded derived categories of the two curves. Recall that $\Phi_{\mathcal{E}}$ is the functor

$$\begin{aligned} \Phi_{\mathcal{E}} : \mathbf{D}(C) &\longrightarrow \mathbf{D}(C'), \\ \mathcal{F} &\longmapsto Rq_*(p^*\mathcal{F} \otimes \mathcal{E}), \end{aligned}$$

where p and q are the projections of $C \times C'$ respectively on C and C' and $\mathcal{E}$ is an object, called the kernel of $\Phi_{\mathcal{E}}$, of the bounded derived category $\mathbf{D}(C \times C')$ of the product of the two curves.

If $\Phi_{\mathcal{E}}$ is an equivalence, it is well known that we get an isomorphism between the two curves. We can ask ourselves if this derived Fourier–Mukai transform (DFM for short) does indeed carry an isomorphism between the Jacobian varieties preserving the principal polarizations.

In order to do that, recall the definition of the Jacobian variety as $\mathrm{Pic}^0(C)$, the degree zero part of the Picard group. What we are actually going to do is to make the DFM descend to an affine map $\Phi_e^P : \mathrm{Pic}_{\mathbb{Q}}(C) \to \mathrm{Pic}_{\mathbb{Q}}(C')$ between the rational Picard groups.

We want to define a morphism $\phi_e^J : J(C) \to J(C')$ compatible with Φ_e^P. This is given in a unique way, since we know how it acts on the degree zero Picard group with rational coefficients. All this is done in Section 2. In Section 3 we answer the main question and we find a correspondence between the Torelli Theorem and the derived characterization of a smooth projective curve.

We are using the following notations: for a smooth projective variety X we denote by $\mathbf{D}(X)$ the bounded derived category of coherent sheaves on X. Given $\mathcal{E}$ in $\mathbf{D}(X)$, we denote by e the class $[\mathcal{E}]$ in $K(X)$. We will use the subscript $_{\mathbb{Q}}$ to mean the tensor product with $\mathbb{Q}$.

2. From the derived Fourier–Mukai to the Jacobian Fourier

In this section, we show how a DFM $\Phi_{\mathcal{E}}$ induces a unique morphism ϕ_e^J between the Jacobian varieties. This is done in three steps. Firstly, we describe the morphism induced by $\Phi_{\mathcal{E}}$ on the rational Picard group. What we find is actually an affine map between rational vector spaces. Secondly, we define a morphism between the Jacobian varieties and we consider its restriction to the Jacobians with rational coefficients, that is $\mathrm{Pic}^0_{\mathbb{Q}}$. This can be done in a unique way. Finally, we show that linearizing the affine map we obtain the map defined by the Fourier transform on Jacobian variety with rational coefficients. The correspondence is functorial.

2.1. *From the derived Fourier–Mukai to an affine map on* $\mathrm{Pic}_{\mathbb{Q}}$

How a DFM between two smooth projective varieties descends at the level of cohomology is well known. A very good reference for that is the recent book by Huybrechts [3].

The first step in making such a descent is going from derived categories to Grothendieck groups. Given C a smooth projective curve, to any object $\mathcal{E}$ in $\mathbf{D}(C)$, we can associate an element $[\mathcal{E}]$ in the Grothendieck group $K(C)$ by the alternate sum of the classes of cohomology sheaves of $\mathcal{E}$. We thus obtain a map [ ] from the isomorphism classes of $\mathbf{D}(C)$ to the Grothendieck group $K(C)$. Given $f: C \to C'$ a projective morphism between smooth projective curves, the pull back $f^*: K(C') \to K(C)$ defines a ring homomorphism. The generalized direct image $f_!: K(C) \to K(C')$, defined by $f_!\mathcal{F} = \sum (-1)^i R^i f_* \mathcal{F}$ for any coherent sheaf $\mathcal{F}$ on C, defines a group homomorphism.

We would like now to define a K-theoretic Fourier–Mukai transform (KFM). Given e a class in $K(C \times C')$, let us define

$$\begin{aligned} \Phi_e^K : K(C) &\longrightarrow K(C'), \\ f &\longmapsto q_!(p^* f \otimes e). \end{aligned}$$

If we are now given a DFM $\Phi_{\mathcal{E}}: \mathbf{D}(C) \to \mathbf{D}(C')$ with kernel $\mathcal{E}$ in $\mathbf{D}(C \times C')$, we obtain the corresponding KFM $\Phi_e^K : K(C) \to K(C')$ by using the K-theoretic kernel $e := [\mathcal{E}]$. By the compatibility of $f_!$ and f^* with [ ], we get the following commutative diagram (see [3], 5.2):

$$\begin{array}{ccc} \mathbf{D}(C) & \xrightarrow{\Phi_{\mathcal{E}}} & \mathbf{D}(C') \\ \big\downarrow {\scriptstyle [\,]} & & \big\downarrow {\scriptstyle [\,]} \\ K(C) & \xrightarrow[\Phi_{[\mathcal{E}]}]{} & K(C'). \end{array} \qquad (1)$$

Consider the exponential Chern character $ch: K(C) \to CH^*_{\mathbb{Q}}(C)$, which maps a class of the Grothendieck group to a cycle in the Chow ring with rational coefficients. For a given $f: C \to C'$ we can define the pull-back $f^*: CH^*_{\mathbb{Q}}(C') \to CH^*_{\mathbb{Q}}(C)$ and the direct image $f_*: CH^*_{\mathbb{Q}}(C) \to CH^*_{\mathbb{Q}}(C')$. In order to get a compatibility with the Chern character ch, the Grothendieck–Riemann–Roch Theorem has to be taken into account.

Theorem 2.1 *(Grothendieck–Riemann–Roch). Let $f: X \to Y$ a projective morphism of smooth projective varieties. Then for any e in $K(X)$*

$$ch\big(f_!(e)\big) = f_*\big(ch(e).Td(f)\big),$$

where $Td(f)$ is the Todd class relative to f.

Now let K_C be the canonical bundle of C and define the affine map

$$\begin{aligned} \Phi_e^P : \mathrm{Pic}_{\mathbb{Q}}(C) &\longrightarrow \mathrm{Pic}_{\mathbb{Q}}(C'), \\ M &\longmapsto q_*\left(p^*M.c_1(e) - \frac{1}{2}c_1(e).p^*K_C + \frac{1}{2}\big(c_1^2(e) + 2c_2(e)\big)\right). \end{aligned}$$

Lemma 2.2. *Let C and C' be smooth projective curves and $\Phi_{\mathcal{E}}:\mathbf{D}(C)\to\mathbf{D}(C')$ be a Fourier–Mukai transform with kernel $\mathcal{E}$ in $\mathbf{D}(C\times C')$. The diagram*

$$\begin{array}{ccc} \mathbf{D}(C) & \xrightarrow{\Phi_{\mathcal{E}}} & \mathbf{D}(C') \\ \downarrow{\scriptstyle c_1\circ[\ ]} & & \downarrow{\scriptstyle c_1\circ[\ ]} \\ \mathrm{Pic}_{\mathbb{Q}}(C) & \xrightarrow[\Phi_e^P]{} & \mathrm{Pic}_{\mathbb{Q}}(C') \end{array} \tag{2}$$

is commutative.

Indeed, let us denote by M both an element of the rational Picard group $\mathrm{Pic}_{\mathbb{Q}}(C)$ and its class in the Grothendieck group $K(C)$. We want to calculate the first Chern class $(ch(\Phi_e^K(M))_1$.

We have the following chain of equalities:

$$\begin{aligned}\big(ch\big(q_!(p^*M\otimes e)\big)\big)_1 &= \left(q_*\left(ch(p^*M\otimes e)\left(1-\frac{1}{2}p^*K_C\right)\right)\right)_1 \\ &= \left(q_*\left(ch(p^*M\otimes e).\left(1-\frac{1}{2}p^*K_C\right)\right)\right)_1 = q_*\left(ch(p^*M).ch(e).\left(1-\frac{1}{2}p^*K_C\right)\right)_2.\end{aligned}$$

Now let us make it more explicit

$$ch(p^*M).ch(e).\left(1-\frac{1}{2}p^*K_C\right) = (1+p^*M).(r+c_1(e)+\frac{1}{2}\big(c_1^2(e)+2c_2(e)\big)\cdot\left(1-\frac{1}{2}p^*K_C\right), \tag{3}$$

where r is the rank of e. We take the degree two part of (3) and we obtain

$$\big(ch\big(q_!(p^*M\otimes e)\big)\big)_1 = p^*M.c_1(e)-\frac{1}{2}c_1(e).p^*K_C+\frac{1}{2}\big(c_1^2(e)+2c_2(e)\big). \tag{4}$$

The morphism Φ_e^P between the Picard groups with rational coefficients commutes with the KFM with kernel e. Combining this with the commutative diagram (1) we get (2).

It is clear that the affine map Φ_e^P restricted to $\mathrm{Pic}^0_{\mathbb{Q}}(C)$ does not give a group morphism to $\mathrm{Pic}^0_{\mathbb{Q}}(C')$. Remark anyway that only the first term of $\Phi_e^P(M)$ depends on M, while the other terms are constant with respect to it.

2.2. *From the Jacobian Fourier to a morphism on* $\mathrm{Pic}^0_{\mathbb{Q}}$

Our aim is to find a morphism between the Jacobian varieties which is compatible with Φ_e^P on the degree zero part of the rational Picard group. Recall that $J(C)=\mathrm{Pic}^0(C)$. Let us define

$$\begin{aligned}\phi_e^J : J(C) &\longrightarrow J(C'), \\ M &\longmapsto q_*\big(p^*(M-\mathcal{O}_C).c_1(e)\big),\end{aligned}$$

where e is a class in the Grothendieck group $K(C\times C')$ and $\mathcal{O}_C$ is the unity in $J(C)$. This is the classical Fourier transform with kernel $c_1(e)$ between the Jacobian varieties, and we are referring to that by JF. There is a unique morphism $\phi_e^{J_{\mathbb{Q}}}: J_{\mathbb{Q}}(C)\to J_{\mathbb{Q}}(C')$ given by restricting ϕ_e^J.

2.3. *They go together*

So far we can say that the DFM with kernel $\mathcal{E}$ uniquely induces on $\mathrm{Pic}_{\mathbb{Q}}(C)$ the morphism Φ_e^P. Linearizing this morphism and restricting it to $\mathrm{Pic}^0_{\mathbb{Q}}(C)$, we get exactly the morphism $\phi_e^{J_{\mathbb{Q}}}$, induced by the JF with kernel $c_1(e)$. We can then conclude that the JF $\phi_e^J: J(C)\to J(C')$ is the only morphism compatible with the DFM $\Phi_{\mathcal{E}}:\mathbf{D}(C)\to\mathbf{D}(C')$.

Remark that, given a smooth projective curve C, the identity on $\mathbf{D}(C)$ is given by the DFM with kernel $\mathcal{O}_\Delta$, the structure sheaf of the diagonal in $C\times C$. The identity on $J(C)$ clearly corresponds to it.

Moreover, by Lemma 2.2 the correspondence between DFMs and the affine maps is functorial. Indeed if we consider two composable DFMs $\Phi_{\mathcal{E}_1}$ and $\Phi_{\mathcal{E}_2}$ and their composition $\Phi_{\mathcal{R}}$, the affine maps $\Phi_{e_1}^P$ and $\Phi_{e_2}^P$ are composable and their composition is given by the affine map Φ_r^P.

The rational linear map $\phi_e^{J_{\mathbb{Q}}}$ is the linearization of Φ_e^P restricted to $\mathrm{Pic}^0_{\mathbb{Q}}$. Consider, in general, two composable affine maps $F_1 : V_1 \to V_2$ and $F_2 : V_2 \to V_3$ between vector spacesand their linearizations f_i. The linearization of the composition $F_2 \circ F_1$ is $f_2 \circ f_1$, the composition of f_1 and f_2. This allows us to state that the correspondence associating the linear map $\phi_e^{J_{\mathbb{Q}}}$ to the DFM $\Phi_{\mathcal{E}}$ is functorial. We can hence state the following:

Lemma 2.3. *The correspondence between derived Fourier–Mukai functors and Fourier transforms on the Jacobian varieties associating ϕ_e^J to $\Phi_{\mathcal{E}}$ is functorial.*

Remark 1. The DFM with kernel $\mathcal{E}[1]$ induces the JF with kernel $-c_1(e)$. This is computed by the definition of $[\ ] : \mathbf{D}(C) \to K(C)$. Here $[1]$ means the shift one step on the right.

The DFM with kernel $\mathcal{E}^\vee$ induces the JF with kernel $-c_1(e)$. This is computed remarking that $c_t(e^\vee) = c_{-t}(e)$.

Given line bundles F on C and F' on C', the DFM with kernel $\mathcal{E} \otimes p^*F \otimes q^*(F')$ induces the JF with kernel $c_1(e)$. This is computed remarking that $p^*F.q^*M = q^*F'.p^*M = p^*F.q^*F' = 0$ for any element M in $J(C)$. In the terminology of [1], Chapter 11, we would say that we have two equivalent correspondences.

3. Preservation of the principal polarization

Let $\Phi_{\mathcal{E}} : \mathbf{D}(C) \to \mathbf{D}(C')$ be a Fourier–Mukai transform. We would like to know under which conditions the morphism $\phi_e^J : J(C) \to J(C')$ preserves the principal polarizations.

Recall that a principal polarization on an Abelian variety A defines an isomorphism $\theta_A : A \to \hat{A}$. Given an isogeny $\phi : A \to B$ between two Abelian varieties, if both A and B have a principal polarization, ϕ respects them if we have a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ {\scriptstyle\theta_A}\downarrow & & \downarrow{\scriptstyle\theta_B} \\ \hat{A} & \xleftarrow[\hat{\phi}]{} & \hat{B}. \end{array}$$

In the case of a smooth projective curve C we know the principal polarization $\theta_C : J(C) \to \hat{J}(C)$. We identify, by means of the isomorphisms θ_C and $\theta_{C'}$ the Jacobian varieties $J(C)$ and $J(C')$ with their duals. We then have to check that the composition $\hat{\phi}_e^J \circ \phi_e^J$ is the identity map on $J(C)$. The dual isomorphism $\hat{\phi}_e^J$ can be obtained as the JF in the opposite way with the same kernel as ϕ_e^J. Namely

$$\begin{aligned} \hat{\phi}_e^J : J(C') &\longrightarrow J(C), \\ M' &\longmapsto p_*\big(q^*(M' - \mathcal{O}_{C'}).c_1(e)\big), \end{aligned} \tag{5}$$

see for example [1], Chapter 11, Proposition 5.3.

Given a DFM $\Phi_{\mathcal{E}} : \mathbf{D}(C) \to \mathbf{D}(C')$, we can describe the kernels of its adjoints. If $\Phi_{\mathcal{E}}$ is an equivalence, its adjoints are its quasi-inverses. The left adjoint is the DFM with kernel

$$\mathcal{E}_L := \mathcal{E}^\vee \otimes q^* K_{C'}[1]. \tag{6}$$

We know by Remark 1 that the JF induced by $\Phi_{\mathcal{E}_L}$ on the Jacobian varieties is given by

$$\begin{aligned} \phi_{e_L}^J : J(C') &\longrightarrow J(C), \\ M' &\longmapsto p_*\big(q^*(M' - \mathcal{O}_{C'}).c_1(e)\big). \end{aligned}$$

Then if $\Phi_{\mathcal{E}}$ induces on the Jacobian varieties the isomorphism ϕ_e^J, its quasi inverse $\Phi_{\mathcal{E}_L}$ induces the dual isomorphism $\hat{\phi}_e^J$. We can state the following by Lemma 2.3:

Theorem 3.1. *Given two smooth projective curves C and C', a Fourier–Mukai functor $\Phi_{\mathcal{E}} : \mathbf{D}(C) \to \mathbf{D}(C')$ is an equivalence if and only if the morphism $\phi_e^J : J(C) \to J(C')$ is an isomorphism preserving principal polarization.*

Recall that for smooth projective curves a derived equivalence always corresponds to an isomorphism (see for example [3], Corollary 5.46). Theorem 3.1 just states the correspondence between the Torelli Theorem (see for example [2], page 359) and the characterization of a curve by its derived category.

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