Interpolation by functions with small spectra

Alexander Olevskii \textsuperscript{a}, Alexander Ulanovskii \textsuperscript{b}

\textsuperscript{a} School of Mathematics, Tel Aviv University, Ramat Aviv, 69978 Israel
\textsuperscript{b} Stavanger University, N-4036 Stavanger, Norway

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Abstract

We show that if \( \Lambda \) is a ‘generic’ separated sequence of reals, then there is an unbounded set \( S \) of arbitrary small measure (union of some neighborhoods of integers) such that every function on \( \Lambda \) with certain decay condition, can be interpolated by an \( L^2 \)-function with the spectrum on \( S \) (Theorem 1). This should be contrasted against results for compact spectra (Theorems 2 and 3).

1. Results

Let \( \Lambda = \{ \cdots < \lambda_{j-1} < \lambda_j < \lambda_{j+1} < \cdots, j \in \mathbb{Z} \} \) be a real sequence. We shall assume that it is separated, i.e. \( \inf_j (\lambda_j - \lambda_{j-1}) > 0 \). By \( D^+(\Lambda) \) we denote the upper uniform density of \( \Lambda \) (see [2, p. 303], [1,3]):

\[
D^+(\Lambda) := \lim_{l \to \infty} \max_{a \in \mathbb{R}} \frac{\#(\Lambda \cap (a, a + l))}{l}.
\]

Given a space of complex sequences \( X = \{ c_j, j \in \mathbb{Z} \} \), we shall say that a set \( S \subset \mathbb{R} \) is an interpolation spectrum for \( X \), if for every \( \{ c_j \} \in X \) there is a function \( F \in L^2(S) \) whose Fourier transform \( \hat{F} \) satisfies:

\[
\hat{F}(\lambda_j) = c_j, \quad j \in \mathbb{Z}.
\]

The case \( X = l^2 \) is classical. Kahane [2] proved that for a single interval \( S \) to be interpolation spectrum, it is necessary that \( \text{mes} S \geq 2\pi D^+(\Lambda) \), and it is sufficient that \( \text{mes} S > 2\pi D^+(\Lambda) \). We mention also Beurling’s result [1].
who proved that the last condition is necessary and sufficient for interpolation of $l^\infty$ by functions bounded on $\mathbb{R}$ with spectra on an interval $S$.

Simple examples show that the sufficient condition above fails already when $S$ is a union of several intervals. However, using a new approach, Landau [3] proved that the necessary condition in Kahane’s result still holds for every bounded set $S$.

In the present note we show that if $S$ is unbounded and $X$ is a space of ‘slowly decreasing sequences’, then no such necessary condition may exist. For ‘generic’ $\Lambda$ we construct interpolation spectra of arbitrary small measure:

**Theorem 1.** Let a separated sequence $\Lambda$ be linearly independent (mod $\pi$) over the field of rational numbers. Then for every $\delta > 0$ there is a set $S$, a union of some of intervals centered at integers, such that:

(i) $\text{mes } S < \delta$;
(ii) for every sequence $c_j = O(|j|^{-\alpha})$, $\alpha > 1$, there is a function $F \in L^2(S)$ satisfying (1).

However, if $S$ is a compact set, an analogue of classical results holds even for spaces $X$ of sequences having a ‘very fast decay’.

In the next result we suppose that the sequence $\Lambda$ is distributed ‘regularly’, i.e. the limit

$$D(\Lambda) := \lim_{l \to \infty} \frac{\#(\Lambda \cap (a, a + l))}{l}$$

exists uniformly with respect to $a$.

**Theorem 2.** Let $S$ be a compact set. If for every sequence $c_j = O(e^{-|j|^\alpha})$, $0 < \alpha < 1$, there exists $F \in L^2(S)$ satisfying (1), then $\text{mes } S \geq 2\pi D(\Lambda)$.

We also prove a version of Landau’s result for ‘interpolation with error’. Denote by $\{e_j, j \in \mathbb{Z}\}$ the standard orthonormal basis in $l^2$.

**Theorem 3.** Let $S$ be a compact set, $\Lambda$ be a separated sequence and $0 < d < 1$. Suppose there is a sequence of functions $F_j \in L^2(S)$, $\sup_j \|F_j\| < \infty$, such that $\|\hat{F}_j|_{\Lambda} - e_j\|_{l^2(\mathbb{Z})} \leq d$ for all $j \in \mathbb{Z}$. Then

$$\text{mes } S \geq 2\pi \left(1 - d^2\right) D^+(\Lambda).$$

The bound (2) is sharp for every $d$.

2. **Proof of Theorem 1**

Here we shall sketch the proof of Theorem 1. It consists of several steps.

1. Without loss of generality we may assume that $\alpha < 2$. Fix any number $\beta$, $1 < \beta < \alpha$. Set

$$S := \bigcup_{j \in \mathbb{Z}} S_j, \quad S_j := (-M_j - 5\gamma_j, -M_j + 5\gamma_j) \cup (M_j - 5\gamma_j, M_j + 5\gamma_j),$$

where

$$\gamma_j := \frac{\gamma}{1 + |j|^\beta},$$

the sequence $M_j$ will be specified in step 4, and $\gamma$ is any small positive number such that $\text{mes } S < \delta$.

2. Set

$$\Lambda_k := (\Lambda - \lambda_k) \setminus \{0\}, \quad k \in \mathbb{Z}.$$ 

The independence condition on $\Lambda$ implies, by Kronecker’s theorem, that for every $N > 0$ the subgroup $\{m\lambda \text{ (mod $\pi$)}, \lambda \in \Lambda_k \cap [-N, N], m \in \mathbb{Z}\}$ is dense in the $l-$dimensional torus, $l$ being the number of elements in $\Lambda_k \cap [-N, N]$. Hence, the $l$ numbers $|\cos(Mx)|, x \in \Lambda_k \cap [-N, N]$, can be made as small as we like by choosing appropriate $M \in \mathbb{N}$.
3. Set
\[ g_j(x) := \cos(M_j(x - \lambda_j)) \left( \frac{\sin \gamma_j(x - \lambda_j)}{\gamma_j(x - \lambda_j)} \right)^5. \]
The spectrum of \( g_j \) belongs to \( S_j \), and we have
\[ g_j(\lambda_j) = 1, \quad (4) \]
and
\[ \|g_j\|_{L^2(\mathbb{R})}^2 \leq \text{const} \cdot (1 + |j|^\beta), \quad j \in \mathbb{Z}. \quad (5) \]

4. By Step 2, the first factor in the definition of \( g_j \) can be made arbitrarily small for \( \lambda \neq \lambda_j, |\lambda - \lambda_j| < N_j \). By using \( N_j \) large enough, one may check that for every positive \( \epsilon > 0 \) there exists a sequence \( M_j \in \mathbb{N} \) such the functions \( g_j \) are small on \( \Lambda \setminus \{\lambda_j\} \) in the sense that
\[ |g_j(\lambda_k)| \leq \frac{\epsilon}{(1 + j^2)(1 + (j - k)^4)}, \quad k \neq j, k, j \in \mathbb{Z}. \quad (6) \]

5. Given a sequence \( \{c_j, j \in \mathbb{Z}\} \), set
\[ \|c\|_\beta^2 := \sum_{j=-\infty}^{\infty} |c_j|^2 (1 + |j|^\beta). \]
Let \( l_\beta^2 \) denote the weighted space of all sequences \( c, \|c\|_\beta < \infty \). Using (6) and (4), one may check that the linear operator \( R \) defined by
\[ R e_j := \sum_{k=-\infty}^{\infty} g_j(\lambda_k)e_k - e_j, \quad j \in \mathbb{Z}, \]
is well defined on \( l_\beta^2 \). Moreover, if \( \epsilon \) in (6) is small enough, the norm of this operator in \( l_\beta^2 \) is less than 1. It follows that the operator \( T := I + R \) is invertible in \( l_\beta^2 \), where \( I \) is the identity operator. We conclude that for every \( c \in l_\beta^2 \) the interpolation problem (1) has a solution \( F \) whose Fourier transform is given by
\[ \hat{F}(x) = \sum_{j \in \mathbb{Z}} b_j g_j(x), \quad \{b_j\} = T^{-1} c \in l_\beta^2. \]

Also, by (3) and (5), we see that \( F \in L^2(S) \).

Remarks.

1. Let \( \xi_j, j \in \mathbb{Z} \), be independent identically distributed random variables having a continuous distribution function concentrated on some neighborhood of the origin. By Theorem 1, the random sequence \( \Lambda = \{n + \xi_n, n \in \mathbb{Z}\} \) has the property that for each \( \delta > 0 \), with probability one there exists a random set \( S, \text{mes} S < \delta \), such that each sequence \( c_j = O(|j|^{-\alpha}), \alpha > 1 \), can be interpolated by an \( L^2 \)-function \( f \) with the spectrum in \( S \).

2. The decay assumption in Theorem 1 cannot be replaced by \( c \in l^2 \). Let \( \Lambda \) be the random sequence above and \( X = l^2 \). Then one can show that with probability one no set \( S, \text{mes} S < 2\pi \), can serve as an interpolation spectrum for \( X \).

3. Compact spectra: interpolation with error

Here we sketch a proof of Theorem 3.

1. Claim: Let \( 0 < c < 1 \) and \( W \) be a linear subspace of the Paley–Wiener space \( PW(-\pi, \pi) \), which is ‘\( c \)-concentrated on some set \( Q \)’ in the sense that
\[ \int_Q |f|^2 > c\|f\|_{L^2(\mathbb{R})}^2, \quad f \in W. \]
Then
\[ \dim W \leq \frac{1}{c} \mes Q + 1. \]

This follows from Landau’s Lemma 1 (compare (iii) and (viii) in [3], p. 41).

2. Fix a small number \( b > 0 \) and set \( S_b := S + (-b, b) \). Let \( \Phi \) be any infinitely smooth function supported on \((-b, b)\) satisfying \( \hat{\Phi}(0) = 1 \) and \( |\hat{\Phi}(x)| < 1, x \neq 0 \). Set
\[ G_j(t) := F_j(t) \ast (e^{-i\lambda_j t} \Phi(t)). \]

Set \( f_j = \hat{F}_j \) and \( g_j = \hat{G}_j \). Clearly, each \( g_j|_A \) approximates \( e_j \) with an \( l^2 \)-error \( \leq d \). One can prove that if \( N \) is sufficiently large, then the space \( Z \) spanned by \( g_j \) when \( |\lambda_j| < N \), is \( c' \)-concentrated on the interval \( J := (1 + b)(-N, N) \), where \( c' \) can be chosen arbitrary close to 1. Hence, for all large \( N \), the space \( Y \) of the inverse Fourier transform of the functions \( g \cdot 1_J, g \in Z \), is \( c \)-concentrated on \( S_b \), again with \( c \) arbitrary close to 1. The claim above, after re-scaling, gives:
\[ \dim Y \leq \frac{(1 + b)N}{\pi c} \frac{1}{\mes S_b + 1}. \]

3. Fix a large number \( N \), and denote by \( \nu = \nu(N) \) the number of points of \( A \) in \((-N, N)\). Define vectors \( v_j \) in the Euclidean space \( \mathbb{C}^\nu \) by
\[ v_j(l) := g_j(\lambda_l), \quad |\lambda_l| < N. \]

Let \( V \) be the linear span of \( v_j \) in \( \mathbb{C}^\nu \). Clearly, \( \dim Y \geq \dim V \). On the other hand, each of \( v_j \) approximates the corresponding \( e_j \) with an error \( \leq d \). A well-known estimate of the Kolmogorov width of octahedron implies
\[ \dim V \geq (1 - d^2) \nu. \]

4. Combining the last three inequalities, one obtains an estimate of \( \nu \). The previous argument can be repeated for each interval \((a - N, a + N)\), uniformly over \( a \). Hence, taking the limit as \( N \to \infty \), we get an estimate of \( D^+(\Lambda) \).

Finally, taking the limit as \( b \to 0 \) and \( c \to 1 \), we obtain (2).

Theorem 2 can be proved basically by the same argument (for regularly distributed \( A \)). Observe that the decay restriction in Theorem 2 can be replaced by any non quasi-analytic one.

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References

