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Differential Geometry

Critical points of the acceleration in $\mathbb{CP}^2(4)$

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Abstract

We study the variational problem associated to the L^2 norm of the angular acceleration for curve variations of constant length. We determine the unit speed closed critical curves with constant slant in $\mathbb{CP}^2(4)$. ***To cite this article: J. Arroyo et al., C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

Points critiques de l'accélération en $\mathbb{CP}^2(4)$. Nous étudions le problème variationnel associé à la norme L^2 de l'accélération angulaire pour des variations de courbe de longueur constante. Nous déterminons les courbes critiques fermées paramétrées par des abscisses curvilignes à pente constante en $\mathbb{CP}^2(4)$. ***Pour citer cet article : J. Arroyo et al., C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Version française abrégée

On suppose que $\gamma : I \to M^n$ est une courbe différentiable régulière dans une variété riemannienne de dimension n. Les lagrangiens qui dépendent des dérivées des coordonnées de la courbe mènent à des problèmes classiques comme, par exemple, la recherche de courbes minimisantes pour la longueur ou pour l'énergie de la courbe. Plus récemment, on a considéré des lagrangiens qui dépendent des dérivées d'ordre supérieur des coordonnées de la courbe dans beaucoup de contextes : la version « particle-like » de la corde rigide, le modèle de la transmutation boson-fermion dans un champ extérieur de Chern–Simons, la théorie de polymères, etc. (voir, par exemple, [4] et les références là-dedans).

Le carré de la norme de l'accélération angulaire peut aussi être considéré dans ce contexte. La première formule variationnelle pour la norme L^2 de l'accélération angulaire $\int_\gamma \langle \frac{D\gamma'}{dt}, \frac{D\gamma'}{dt} \rangle \, dt$ était calculée dans [5], tandis que la deuxième formule de variation et des aspects liés comme des champs de Jacobi, la forme de l'indice, etc., étaient étudiés dans [3]. Nous sommes intéressés à l'étude des points critiques de l'accélération pour des variations de courbe à longueur constante, $\mathcal{F}(\gamma) = \int_0^L (\langle \frac{D\gamma'}{dt}, \frac{D\gamma'}{dt} \rangle + \lambda \|\gamma'\|) \, dt$, $\lambda \in \mathbb{R}$. La recherche de ce modèle dans le calcul variation-

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nel classique donne lieu à des équations différentielles non linéaires d'ordre 4 très compliquées. Dans [1] nous avons utilisé une approximation géométrique pour obtenir la détermination analytique explicite de ces courbes critiques paramétrées à vitesse unité dans espaces réels à courbure constante, donc il semble normal de prolonger cette analyse aux les espaces complexes à courbure holomorphe constante. Dans cette Note, nous considérons les points critiques paramétrées à vitesse unité de $\mathcal{F}$ dans $\mathbb{CP}^2(4)$ qui vérifient de plus la propriété géométrique suivante : l'angle entre le plan tangente complexe et le plan osculateur est constante tout le long de γ. Nous appelons ces courbes courbes *slant* dans $\mathbb{CP}^2(4)$. La version invariant de cette énergie correspond au problème classique d'elastica de D. Bernoulli et L. Euler [2].

D'autre part, Barros, Singer et le deuxième auteur ont introduit dans [2] une *repère complexe* tout le long d'une courbe γ qui donne lieu à des *courbures complexes*. Quelques propriétés de ces courbures, spécialement en liaison avec les courbes *slant* dans $\mathbb{CP}^2(4)$, ont été prouvées dans [2] et ont été révisées dans §2 pour utilisation postérieure.

Dans §3 nous nous limitons à des points critiques fermés paramétrés à vitesse unité. Dans ce cas, nous voyons que les équations différentielles non linéaires d'Euler–Lagrange (11) peuvent bien être exprimées en termes des courbures de Frenet (13) et, avec une plus grande symétrie, en termes des courbures complexes (14). Ceci est utilisé d'abord pour montrer qu'un point critique paramétré à vitesse unité doit être une hélice de Frenet (Proposition 3.1) et, alors, pour calculer leur courbures de Frenet (17), (18). De plus, combinant ces résultats avec le fait que chaque courbe *slant* dans $\mathbb{CP}^2(4)$ est l'image sous la projection naturelle d'un groupe à un paramètre (voir §2), nous sommes capables de déterminer les courbes critiques *slant* fermées dans le Théoréme 3.2. En particulier, si $\lambda = 0$, les points critiques du carré de la norme de l'accélération angulaire total sont connus comme cubiques de Riemannian et ils peuvent être obtenus comme conséquence du susdit résultat (Corollaire 3.3).

1. Introduction

Assume that $\gamma : I \to M^n$ is a smooth regular curve in an n-dimensional Riemannian manifold. Lagrangians depending on the derivatives of the curve coordinates lead to classical problems as, for instance, the searching of minimizing curves for the length or for the curve energy integral. More recently, Lagrangians depending on higher derivatives of the curve coordinates have been considered in many contexts: the particle-like version of the rigid string, the model of boson-fermion transmutation in external Chern–Simons field, the polymer theory, etc. (see, for instance, [4] and the references therein).

The total squared norm of the angular acceleration $\mathcal{F}_o(\gamma) = \int_\gamma \langle \frac{D\gamma'}{dt}, \frac{D\gamma'}{dt} \rangle \, dt$ can also be considered in this context. Its critical points, a generalization of cubic splines, are called *Riemannian cubics*. Here, we are interested in the study of the critical points of the acceleration for curve variations with constant length, that is critical points of $\mathcal{F}(\gamma) = \int_0^L (\langle \frac{D\gamma'}{dt}, \frac{D\gamma'}{dt} \rangle + \lambda \|\gamma'\|) \, dt$, $\lambda \in \mathbb{R}$. The invariant version of this energy corresponds to the *elasticae* problem as introduced by D. Bernoulli and L. Euler which has been intensively studied in the literature [2]. If we put $\|\gamma'\|^2$ instead of $\|\gamma'\|$ in $\mathcal{F}$, then critical curves are known as *Riemannian splines in tension*. They include *Riemannian cubics* as special case and are used in non-linear dynamic interpolation problems on Riemannian manifolds with applications in robotics, engineering, geometric design and rigid-body trajectory planning (see [3,5,6]). Investigation of $\mathcal{F}$ in the classical variational calculus gives rise to very complicated non-linear differential equations of order 4. In [1] we have used a geometric approach to obtain the explicit analytical determination of its unit speed critical curves in real space forms, so it seems natural to extend this analysis to complex space forms. In this Note, we consider the unit speed critical curves of $\mathcal{F}$ in the complex projective plane $\mathbb{CP}^2(4)$, which satisfy in addition the following geometric property: the angle between the complex tangent plane and the osculating plane is constant along γ (*slant* curves).

On the other hand, Barros, Singer and the second author have introduced in [2] a *Complex reference frame* along a curve γ which gives rise to *complex curvatures*. Some properties of these curvatures, specially in connection with *slant* curves in $\mathbb{CP}^2(4)$, were proven in [2] and are reviewed in §2 for later use.

In §3 we restrict ourselves to unit speed closed critical points. In this case, we see that the Euler–Lagrange equations (11) can be expressed nicely in terms of the Frenet curvatures (13) and, more symmetrically, in terms of the complex curvatures (14). These are used first to show that a unit speed slant critical point must be a Frenet helix (Proposition 3.1) and, then, to compute their Frenet curvatures (17), (18). Moreover, combining these results with the fact that every slant helix in $\mathbb{CP}^2(4)$ is the image under the natural projection of a one-parameter group (see §2), we are able to determine the closed slant critical curves in Theorem 3.2. In particular, closed Riemannian cubics are obtained in Corollary 3.3.

2. Slant Frenet curves

(For details in this section see [2].) Let $\mathbb{CP}^2(4)$ be the 2-dimensional complex projective space of constant holomorphic sectional curvature 4, endowed with complex structure J, Fubini–Study metric $\{,\}$, Levi-Civita connection ∇ and curvature tensor R. Given a C^∞-immersed curve $\gamma(t)$, $\gamma:[0,L]\to\mathbb{CP}^2(4)$, we denote by $\{T(s),\xi_2(s),\xi_3(s),\xi_4(s)\}$ the Frenet frame along γ, where s is the arclength parameter and $T(s)=\gamma'(s)$ is the unit tangent vector field. The standard Frenet equations are $\nabla_T T=\kappa\xi_2$, $\nabla_T\xi_2=-\kappa T+\tau\xi_3$, $\nabla_T\xi_3=-\tau\xi_2+\delta\xi_4$, $\nabla_T\xi_4=-\delta\xi_3$, where $\{\kappa,\tau,\delta\}$ are the Frenet curvatures. A curve γ is said to be a *helix* if $\{\kappa,\tau,\delta\}$ are constant. We put $JT=A_2\xi_2+A_3\xi_3+A_4\xi_4$, where $A_i(s)=\cos\phi_i(s)=\langle JT,\xi_i\rangle$, $0\leqslant\phi_i\leqslant\frac{\pi}{2}$, $2\leqslant i\leqslant 4$, and $\sum_{i=2}^{i=4}A_i^2=1$. A *complex Frenet frame* on γ can be constructed as follows. Let us denote by $\Pi:\mathbb{S}^5(1)\to\mathbb{CP}^2(4)$ the Hopf map and let $\pi:\mathbb{SU}(3)\to\mathbb{CP}^2(4)$ be the canonical projection. First, we lift the curve γ to a horizontal curve $Y(s)$ in $\mathbb{S}^5(1)$ via Π. Different horizontal lifts differ by a factor of the form e^{ri}, r being a constant. Since Π is a Riemannian submersion, the tangent vector $T(s)$ lifts to a unit vector $\bar T(s)=Y'(s)$. Now we may uniquely choose a vector U orthogonal to T so that its horizontal lift $\bar U$ gives the third vector in a special unitary frame $\sigma(s)=\{Y,\bar T,\bar U\}$ in $\mathbb{C}^3$. Hence, $\sigma(s)$ is a lifting of the curve $\gamma(s)$ to a curve in $\mathbb{SU}(3)$. By projecting down $\sigma(s)$ via Π, we obtain a new frame $\{\Pi_*(\bar T)=T,\ \Pi_*(i\bar T)=JT,\ \Pi_*(\bar U)=U,\ \Pi_*(i\bar U)=JU\}$ on $\gamma(s)$ satisfying

$$\nabla_T T=\kappa_1 JT+\kappa_2 U+\kappa_3 JU,\qquad \nabla_T JT=-\kappa_1 T-\kappa_3 U+\kappa_2 JU,\tag{1}$$

$$\nabla_T U=-\kappa_2 T+\kappa_3 JT-\kappa_1 JU,\qquad \nabla_T JU=-\kappa_3 T-\kappa_2 JT+\kappa_1 U,\tag{2}$$

$$\kappa_1^2+\kappa_2^2+\kappa_3^2=\kappa^2,\qquad \kappa A_2=\kappa\cos\phi_2=\kappa_1,\tag{3}$$

$$(\kappa_1')^2+(\kappa_2')^2+(\kappa_3')^2=(\kappa')^2+\kappa^2\tau^2,\qquad \kappa' A_2+\kappa\tau A_3=\kappa_1'.\tag{4}$$

Here $()'$ denotes derivative with respect to the arclength parameter s. A curve γ of $\mathbb{CP}^2(4)$ is said to be a *slant curve* if $A_2=\langle JT,\xi_2\rangle=\cos\phi_2$ is constant along γ. A slant curve is *proper* if $A_2\in(0,1)$. Then a unit speed slant curve with non-zero torsion must satisfy

$$A_2'=\tau A_3,\qquad (\kappa+\delta)\sqrt{1-A_2^2}=A_2\tau.\tag{5}$$

Assume now that γ is a proper slant Frenet helix. Let us denote ϕ_2 simply by ϕ. Then we can find a function $\psi(s)=w\cdot s$ such that

$$\kappa_1=\kappa\cos\phi,\quad \kappa_2=\kappa\sin\phi\cos\psi(s),\quad \kappa_3=\kappa\sin\phi\sin\psi(s).\tag{6}$$

Moreover, it was also proved in [2] that a proper slant Frenet helix of $\mathbb{CP}^2(4)$ is the image of a one-parameter subgroup of $\mathbb{SU}(3)$ under the natural projection π, $\bar\sigma(s)=\bar\sigma(0)\mathrm{e}^{Ms}$, where $M\in\mathfrak{su}(3)$ is given by

$$M=\begin{pmatrix}-\frac{i\omega}{3} & -1 & 0\\ 1 & (C-\frac{\omega}{3})i & -S\\ 0 & S & (\frac{2\omega}{3}-C)i\end{pmatrix},\tag{7}$$

and $C=\kappa_1$ and S are constants satisfying

$$\kappa_1=\kappa\cos\phi,\quad S=\kappa\sin\phi=\kappa\sqrt{1-A_2^2},\quad \kappa_2=S\cos(\omega s),\quad \kappa_3=S\sin(\omega s),\quad \kappa^2=C^2+S^2.\tag{8}$$

3. Critical slant curves

We consider $\widetilde{\Omega}$ the space of C^∞ immersed closed curves in $\mathbb{CP}^2(4)$ defined in $[0,L]$ and denote by γ' the tangent field of γ. For a fixed $\lambda\in\mathbb{R}$, let us take the following energy functional $\mathcal{F}:\widetilde{\Omega}\to\mathbb{R}$

$$\mathcal{F}(\gamma)=\int_0^L\left(\left\langle\frac{D\gamma'}{dt},\frac{D\gamma'}{dt}\right\rangle+\lambda\|\gamma'\|\right)dt.\tag{9}$$

Given $\gamma \in \widetilde{\Omega}$, we consider a C^∞ variation $\Gamma : (-\epsilon, \epsilon) \times [0, L] \to \mathbb{CP}^2(4)$ such that $\Gamma(0,t) = \gamma(t)$ and $\Gamma(u,t) = \gamma_u(t) \in \widetilde{\Omega}$. Combining the well known first variation formula for the length with the corresponding one for the L^2 norm of the acceleration [5], one can compute the first variation formula of (9)

$$\frac{1}{2}\frac{d\mathcal{F}}{du}(0) = \int_0^L \langle \mathcal{E}(\gamma), W\rangle \, dt + \mathcal{B}[W, \gamma'], \tag{10}$$

where $W = \frac{\partial \Gamma}{\partial u}(0,t)$ is the variation field of Γ, and the Euler–Lagrange and boundary operators are given respectively by

$$\mathcal{E}(\gamma) = \frac{D^3\gamma'}{dt^3}(t) + R\left(\frac{D\gamma'}{dt}(t), \gamma'(t)\right)\gamma'(t) - \lambda \frac{D}{dt}\left(\frac{\gamma'(t)}{\|\gamma'(t)\|}\right), \tag{11}$$

$$\mathcal{B}[W, \gamma'] \,\Big|_0^L = \left\langle \frac{DW}{dt}(t), \frac{D\gamma'}{dt}(t)\right\rangle \Big|_0^L - \left\langle W(t), \frac{D^2\gamma'}{dt^2}(t)\right\rangle \Big|_0^L + \lambda \left\langle W(t), \frac{\gamma'}{\|\gamma'\|}(t)\right\rangle \Big|_0^L. \tag{12}$$

However, $\gamma \in \widetilde{\Omega}$ implies that $\mathcal{B}[W, \gamma']|_0^L = 0$. Therefore, a closed curve is a critical point of $\mathcal{F}$, if and only if, $\mathcal{E}(\gamma) = 0$. One sees from (11) that *every closed geodesic is a critical point of* $\mathcal{F}$. From now on, *we assume that* $\gamma \in \widetilde{\Omega}$ *is a non-geodesic unit speed curve which is critical point of* $\mathcal{F}$. Then, combining the Frenet equations and $\mathcal{E}(\gamma) = 0$, we have first that κ must be constant and also that

$$3A_2^2 = \kappa^2 + \tau^2 + \lambda - 1, \quad 0 = \tau' + 3A_2A_3, \quad 0 = \tau\delta + 3A_2A_4, \tag{13}$$

where $A_i = \langle JT, \xi_i\rangle$, $2 \leqslant i \leqslant 4$, and $()'$ denotes derivative with respect to the arclength parameter s. Analogously, combining $\mathcal{E}(\gamma) = 0$ and Eqs. (1) and (2) gives

$$0 = \kappa_i'' + \kappa_i(\varepsilon_i - \kappa^2 - \lambda) + \kappa_j\kappa_h' - \kappa_h\kappa_j', \tag{14}$$

where $i \in \{1,2,3\}$, $j \neq i \neq h$, $j < h$ if $i = 2$, $j > h$ if $i \neq 2$ and $\varepsilon_1 = 4$, $\varepsilon_2 = \varepsilon_3 = 1$. It is not difficult to see that τ vanishes identically if and only if, either γ is a circle of curvature $\kappa = \sqrt{4-\lambda}$ in $\mathbb{S}^2(4)$ or a circle of curvature $\kappa = \sqrt{1-\lambda}$ in $\mathbb{RP}^2(1)$. Therefore, we only have to take care of the case $\tau \neq 0$. Then we have

Proposition 3.1. *Let* $\gamma \in \widetilde{\Omega}$ *be a unit speed critical curve of* $\mathcal{F}$ *with non-zero torsion in* $\mathbb{CP}^2(4)$. *Then* γ *is a proper slant curve, if and only if,* γ *is a Frenet helix.*

Proof. If γ is a helix, then κ, τ are constant and (13) implies that A_2 is constant. Conversely, if γ is a proper slant curve with $\tau \neq 0$, then we have from (5) that $A_3 = 0$, thus $A_4 = \sqrt{1 - A_2^2}$ is also non-zero constant. Since γ is a critical point, then κ is constant and (13) implies that τ and δ are constant also. □

Given $0 \leqslant \lambda < 1$ we use for simplicity the following notation: $\alpha_1 = -\frac{\sqrt{1-\lambda}(8+\lambda)}{(4-\lambda)^{3/2}}$, $\alpha_2 = \frac{\sqrt{1-\lambda}(8+\lambda)}{(4-\lambda)^{3/2}}$, $\vartheta_1 = \frac{\arccos(\alpha_1)}{3}$, $\vartheta_2 = \frac{\arccos(\alpha_2)}{3}$, $\omega_1 = -\sqrt{1-\lambda}$, $\omega_2 = \sqrt{1-\lambda}$ and $\mho(x) = \frac{9\lambda x + 8x^3}{(6-3\lambda-2x^2)^{3/2}}$. Then, we can prove the following:

Theorem 3.2. *If* $\mathcal{F}$ *admits a non-trivial unit speed slant critical curve* $\gamma \in \widetilde{\Omega}$, *then* $0 \leqslant \lambda < 1$. *Now, fix a number* $0 \leqslant \lambda < 1$ *and let us choose a rational number* q *such that* $\sqrt{3}\tan(\vartheta_2) < q < \sqrt{3}\tan(\vartheta_1)$. *Define an angle* θ, $\vartheta_2 < \frac{\theta}{3} < \vartheta_1$, *by* $q = \sqrt{3}\tan\frac{\theta}{3}$ *and take* ω *as the only solution of* $\cos\theta = \mho(\omega)$ *in* (ω_1, ω_2). *Then the unit speed helix of* $\mathbb{CP}^2(4)$ *determined by the last equation of* (16) *and by* (17), (18), *is a closed slant curve critical for* $\mathcal{F}$ *in* $\mathbb{CP}^2(4)$. *Its slant angle is given by* (17) *and satisfies* $A_2^2 < (\frac{-3+\sqrt{3}\sqrt{4-\lambda}}{3})^2$. *Any unit speed closed slant curve critical for* $\mathcal{F}$ *in* $\mathbb{CP}^2(4)$ *with* $\tau \neq 0$ *can be obtained in this way.*

Proof. Let us assume that $\gamma \in \widetilde{\Omega}$ is a proper slant critical curve of $\mathcal{F}$. Then, by Proposition 3.1, γ is a non-zero torsion helix with constant slant angle $\cos\phi_2 = A_2$. Now, we want to compute its curvatures. We use (5) and (13) to obtain

$$\tau^2 = \frac{A_2^2 - 1}{2}\theta(\lambda, A_2), \quad \kappa^2 = 3A_2^2 - \tau^2 - \lambda + 1, \quad \delta = -3\,\frac{A_2 A_4}{\tau}, \tag{15}$$

where $\theta(\lambda, A_2) = 3A_2^2 - 1 + \lambda + 3\sqrt{(A_2^2 - b_1^2)(A_2^2 - b_2^2)}$ and $b_1 = \frac{3+\sqrt{3}\sqrt{4-\lambda}}{3}$, $b_2 = \frac{-3+\sqrt{3}\sqrt{4-\lambda}}{3}$. Hence (15) implies that $A_2^2 < (\frac{-3+\sqrt{3}\sqrt{4-\lambda}}{3})^2$. Following the arguments of §2, we can find two constants C and S satisfying (8). Since γ is critical, we may substitute them into (14) and

$$3C = (C^2 + S^2)\omega - C\omega^2, \quad \omega^2 = 1 - \lambda + C\omega - (C^2 + S^2), \quad C = \frac{-w(\lambda - 1 + w^2)}{3}. \tag{16}$$

By using (8), (15) and (16), we get that the other constants can also be expressed in terms of ω:

$$\kappa^2 = \frac{(3 + w^2)(1 - \lambda - w^2)}{3}, \quad A_2^2 = \frac{w^2(1 - \lambda - w^2)}{3(3 + w^2)}, \quad \delta^2 = \frac{3(1 - \lambda - w^2)}{3 + w^2}, \tag{17}$$

$$S^2 = \frac{(1 - \lambda - w^2)(9 + (2 + \lambda)w^2 + w^4)}{9}, \quad \tau^2 = \frac{w^2(9 + (2 + \lambda)\,w^2 + w^4)}{3(3 + w^2)}. \tag{18}$$

These equations imply that $\lambda < 1$, and $\omega^2 < 1 - \lambda$. Observe that everything is valid so far for curves with prescribed initial and final points as well as initial and final velocities. On the other hand, we know that $\gamma(s)$ is the image of a one-parameter subgroup of $\mathbb{SU}(3)$. Thus, to determine the closed critical helices it is enough to find out the condition for the lifted curve $\bar{\sigma}(s)$ to be a closed geodesic in $\mathbb{SU}(3)$. In order to do so, we must find a positive number, say ρ, so that $\bar{\sigma}(s + \rho) = \bar{\sigma}(s)$. Since $\bar{\sigma}(s) = \mathrm{e}^{sM}$ where M is given in (7), this reduces to the problem of finding ρ such that the eigenvalues of $\rho \cdot M$ are all integer multiples of $2\pi i$. Let η_1, η_2, η_3 be the eigenvalues of M. Since M is in $\mathfrak{su}(3)$, we have $\eta_1 + \eta_2 + \eta_3 = 0$. It follows that the required condition is that $\frac{\eta_2}{\eta_1}$ be rational. Moreover, by using (16) we see that the characteristic polynomial of M is

$$\chi_M(\eta) = \eta^3 + \frac{3\omega + 9C + \omega^3}{3\omega}\eta + i\,\frac{(18\omega - 54C - 2\omega^3)\omega}{27} = 0. \tag{19}$$

Replacing η by ir and using (16), the above polynomial equation reduces to $r^3 + d\,r + e = 0$, where $d = -2 + \lambda + \frac{2\,w^2}{3}$, $e = \frac{-2\,\lambda\, w}{3} - \frac{16\,w^3}{27}$. Hence (19) has either one or three pure imaginary roots. The discriminant of this polynomial $\Delta = \frac{e^2}{4} + \frac{d^3}{27}$ is given by

$$\Delta = \frac{(\lambda - 2)^3}{27} + \frac{8 - 8\lambda + 5\lambda^2}{27}\omega^2 + \frac{4(5\lambda - 2)}{81}\omega^4 + \frac{8}{81}\omega^6. \tag{20}$$

It is easy to prove that $\Delta < 0$ and then it has three real roots which are given by $r_1 = a\cos(\frac{\theta}{3})$, $r_2 = a\cos(\frac{\theta + 2\pi}{3})$ and $r_3 = a\cos(\frac{\theta + 4\pi}{3})$, where $a = \sqrt{-d/3}$, $\cos\theta = -\frac{e}{2a^3}$ and d, e are defined in the previous paragraph. In particular $a = \frac{\sqrt{6 - 3\lambda - 2w^2}}{3}$ and $\cos\theta = \mho(\omega)$. Hence the critical helix is closed if $\frac{\eta_2}{\eta_1} = \frac{-1}{2} + \frac{1}{2}\sqrt{3}\tan\frac{\theta}{3}$ is rational. One can prove that $\cos\theta$ increases monotonically for $0 \leqslant \lambda < 1$, $\omega^2 < 1 - \lambda$. Also

$$\lim_{\omega \to -(\sqrt{1-\lambda})^+} \cos\theta = -\frac{\sqrt{1 - \lambda}(8 + \lambda)}{(4 - \lambda)^{3/2}}, \qquad \lim_{\omega \to (\sqrt{1-\lambda})^-} \cos\theta = \frac{\sqrt{1 - \lambda}(8 + \lambda)}{(4 - \lambda)^{3/2}}.$$

Therefore, $\cos\theta$ increases monotonically from α_1 to α_2 for $-\sqrt{1 - \lambda} < \omega < \sqrt{1 - \lambda}$. This concludes the proof. □

In particular, if $\lambda = 0$ critical points of $\mathcal{F}$ are known as Riemannian cubics [5] and we have:

Corollary 3.3. *Let us choose a rational number q such that $0 < q < 3$. Define an angle θ, $0 < \theta < \pi$, by $q = \sqrt{3}\tan\frac{\theta}{3}$ and take ω as the only solution of $\cos\theta = \frac{8\omega}{(6 - 2\omega^2)^{3/2}}$ in $(-1, 1)$. Then the unit speed helix of $\mathbb{CP}^2(4)$ determined by the last equation of (16) and by (17), (18), is a closed slant cubic in $\mathbb{CP}^2(4)$. Its slant angle is given by (17) and satisfies $A_2 < \frac{2}{\sqrt{3}} - 1$. Any unit speed closed slant cubic of $\mathbb{CP}^2(4)$ with $\tau \neq 0$ can be obtained in this way.*

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