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Dynamical Systems

Expanding cocycles for interval maps

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Abstract

We give a cocycle expansivity result for C^2 multimodal interval maps with non-flat critical points. It extends the Mañé hyperbolicity theorem to also describe orbits which pass near critical points. ***To cite this article: N. Dobbs, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

Cocycles dilatants pour des transformations de l'intervalle. On étend le théorème d'hyperbolicité de Mañé pour traiter des orbites qui passent par des voisinages critiques pour des applications multimodales de l'intervalle. On démontre que, pour des cocycles bien adaptés, ces applications sont dilatantes. ***Pour citer cet article : N. Dobbs, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Version française abrégée

Pour les systèmes dynamiques réels en dimension un, le théorème d'hyperbolicité de Mañe [3] est d'une grande importance. Il dit que la dynamique est hyperbolique et donc se comporte bien, tant qu'on est loin des points critiques et des orbites périodiques paraboliques ou attractives.

Notre résultat décrit à la fois la dynamique hyperbolique et la dynamique non-hyperbolique. Il décrit ainsi une propriété des orbites qui passent arbitrairement proches des points critiques. On démontre que les applications sont dilatantes pour des cocycles que l'on définit dans cet article.

Soit f une transformation de l'intervalle I de classe C^2. Notons Df la dérivée de f. On dit que f a un point critique en c, ou $c \in \mathrm{Crit}(f)$, si $Df(c) = 0$. On suppose que la cardinalité de $\mathrm{Crit}(f)$ est finie, et que, pour chaque $c \in \mathrm{Crit}(f)$, il existe un difféomorphisme ψ de classe C^2, un $\beta > 1$ et un voisinage de c sur lequel $f(x) = \pm|\psi(x)|^\beta + f(c)$.

On définit $D_{\delta,W} f$ par les équations (2) et (3) et $D_{\delta,W} f^n(x)$ par la relation de récurrence $D_{\delta,W} f^n(x) = D_{\delta,W} f^{n-1}(f(x)) \cdot D_{\delta,W} f(x)$. Soit $\mathcal{A}$ la réunion des bassins d'attraction immédiats des attracteurs périodiques et soit $\mathcal{P}$ un voisinage des points périodiques paraboliques.

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Théorème 0.1. *Pour chaque $\delta > 0$, il existe des voisinages arbitrairement petits W de* $\mathrm{Crit}(f)$ *et des nombres $N_0 \geqslant 1$, $\lambda > 1$ tels que pour $n \geqslant N_0$*

$$\left|D_{\delta,W} f^n(x)\right| \geqslant \lambda > 1$$

pour tout x tel que, pour tout i avec $0 \leqslant i \leqslant n$, $f^i(x) \notin \mathcal{A} \cup \mathcal{P}$.

Ce problème s'est posé en étudiant la question importante de conjugaison des applications lisses avec des applications à dérivée schwarzienne négative. Cette question a été récemment résolue par Graczyk et Sands (en préparation), qui utilisent une version de notre théorème. Le problème est intéréssant en soi.

Corollaire 0.2. *Le théorème reste vrai si on prend $W = \{x \in I\colon |Df(x)| < \delta\}$.*

Démonstration. D'après le théorème, il existe $N_0 \geqslant 1$, $\lambda > 1$ et un petit voisinage $W' \subset W$ tels que, pour $n \geqslant N_0$, $|D_{\delta,W'} f^n(x)| \geqslant \lambda > 1$. Si $W \supset W'$ alors $|D_{\delta,W} f(x)| \geqslant |D_{\delta,W'} f(x)|$, donc $|D_{\delta,W} f^n(x)| \geqslant \lambda > 1$, ce qu'il fallait démontrer. □

On va esquisser la démonstration du théorème, laissant les démonstrations rigoureuses pour la partie en anglais qui suit.

Soient $U \subset I$ et $r(x) = \inf\{n \geqslant 1\colon f^n(x) \in U\}$. Alors l'application de premier retour ϕ_U pour un ensemble U est définie, à l'endroit où $r(x)$ l'est, par $\phi_U(x) = f^{r(x)}(x)$. Un intervalle U est dit *à retour régulier* si $f^n(\partial U) \cap U = \emptyset$ pour tout $n > 0$.

On fixe un voisinage W de $\mathrm{Crit}(f)$ tel que $|D_{\delta,W} f(x)| > 4e|Df(x)|$ pour tout $x \in W$. Puis on démontre la proposition centrale :

Proposition 0.3. *Pour chaque $\lambda > 1$, pour chaque point critique $c \in \mathrm{Crit}(f) \setminus \mathcal{A}$ il existe un voisinage V_* de c tel que l'application de premier retour ϕ_{V_*} satisfait $|D_{\delta,W} \phi_{V_*}| \geqslant \lambda$.*

Ceci est assez délicat. Utilisant un résultat profond, les *bornes a priori* de van Strien et Vargas [4], il existe $\alpha > 0$ et des voisinages arbitrairement petits $V_* \subset V_c$ de c avec les propriétés suivantes. V_c est un intervalle à retour régulier. Soit $\mathrm{dist}(V_*, \partial V_c) \geqslant \alpha|V_*|$, soit V_c est la composante connexe du domaine de définition de ϕ_{V_c} contenant c, $V_* = V_c$ et $|\phi_{V_c}(V_c)| \geqslant \alpha|V_c|$.

Soit x dans le domaine de ϕ_{V_*} et prenons $n > 0$ minimal tel que $\phi^n_{V_c}(x) \in V_*$. Soit U la composante connexe du domaine de définition de $\phi^n_{V_c}$ qui contient x ; alors $|\phi^n_{V_c}(U)| \geqslant \alpha|U|$. Donc il existe $y \in U$ tel que $|D\phi^n_{V_c}(y)| \geqslant \alpha$. En rétrécissant les voisinages critiques, des arguments de contrôle de distorsion permettent d'obtenir la dilatation du cocycle.

Ensuite, on fait un peu d'analyse combinatoire. Il y a deux cas. Dans le premier cas, l'orbite d'un point passe souvent près des voisinages des points critiques. Alors il y a un segment de l'orbite qui passe beaucoup de fois près d'un point critique particulier. La proposition implique que le cocycle le long du segment est très dilaté. Dans le deuxième cas, il y a un segment d'orbite très long qui évite les voisinages des points critiques ainsi que $\mathcal{A} \cup \mathcal{P}$. D'après le théorème d'hyperbolicité de Mañé, le cocycle le long de ce segment est exponentiellement dilaté.

Ayant trouvé un segment d'orbite pour lequel le cocycle est très dilaté, il suffit maintenant de minorer le cocycle le long de chacun des deux segments d'orbite qui restent. Donné un tel segment, on le découpe en des segments ayant l'un des trois types suivants :

- Un segment d'orbite ayant ses extremités dans un même voisinage critique. La Proposition 0.3 montre que le cocycle est dilaté.
- Un segment d'orbite qui évite les voisinages critiques et $\mathcal{A} \cup \mathcal{P}$, assez long pour appliquer le théorème de Mañé. D'après ce théorème le cocycle est dilaté.
- Un segment de longueur borné uniformément. Le cocycle est minoré.

On borne le nombre de segments du troisième type par récurrence sur le nombre de voisinages différents des points critiques visités par un segment d'orbite. Le résultat en découle.

1. Introduction

The Mañe Hyperbolicity Theorem [3] is central to smooth one-dimensional discrete dynamical systems, implying that the dynamics, disjoint from critical neighbourhoods and from a neighbourhood of the attracting and parabolic periodic orbits, is hyperbolic (and thus straightforward).

Our result covers both the hyperbolic and the more interesting non-hyperbolic dynamics, extending Mañé's theorem to also cover orbits which pass arbitrarily close to the critical set. We define appropriate derivative cocycles and show that these are expanding.

Let f be a C^2 map of the interval I to itself with a finite number of critical points denoted $\mathrm{Crit}(f)$. We denote the derivative of f by Df and say a point is *critical* at c if $Df(c)=0$. We suppose the critical points are non-flat, that is for x near $c \in \mathrm{Crit}(f)$, we can write

$$f(x) = \pm\left|\psi(x)\right|^{\beta} + f(c), \tag{1}$$

where ψ is a C^2 diffeomorphism, $\psi(c)=0$ and $\beta > 1$.

We define a δ-W derivative cocycle $D_{\delta,W}$ for f as follows:

$$D_{\delta,W} f(x) = \delta, \quad x \in W, \tag{2}$$

$$D_{\delta,W} f(x) = Df(x), \quad x \notin W. \tag{3}$$

$D_{\delta,W} f^n(x)$ is defined inductively as $D_{\delta,W} f^{n-1}(f(x)) \cdot D_{\delta,W} f(x)$. Denote by $\mathcal{A}$ the union of immediate basins of attraction of periodic attractors and by $\mathcal{P}$ some small neighbourhood of the parabolic periodic points (q-periodic points p such that $|Df^q(p)|=1$). The following is the main result of the Note:

Main Theorem 1. *Given any $\delta > 0$ there exist arbitrarily small neighbourhoods W of $\mathrm{Crit}(f)$ and $N_0 \geqslant 1, \lambda > 1$ such that for each $n \geqslant N_0$*

$$\left|D_{\delta,W} f^n(x)\right| \geqslant \lambda > 1$$

for every x satisfying $f^i(x) \notin \mathcal{A} \cup \mathcal{P}$ for all $0 \leqslant i \leqslant n$.

This problem arose while studying the question of conjugacy of smooth interval maps with maps having negative Schwarzian derivative. The problem seems interesting in and of itself. The conjugacy question has been solved recently by Graczyk et Sands (in preparation), who use a version of this theorem.

One may like to compare this result with a related (simplified) statement which holds for Misiurewicz maps:

Fact 1.1 *(van Strien, [1] theorem III.6.3). Let f be a C^2 map of the interval I to itself with a finite critical set $\mathrm{Crit}(f)$ and non-flat critical points. Suppose there is a neighbourhood U of $\mathrm{Crit}(f)$ such that $f^n(\mathrm{Crit}(f)) \cap U = \emptyset$ for all $n > 0$. Suppose also that all periodic points of f are hyperbolic repelling. Then for each sufficiently small neighbourhood W of $\mathrm{Crit}(f)$ there exist constants $\lambda > 1, C > 0$ such that for each $x \in I$ and each $k \geqslant 1$:*

$$\left|Df^k(x)\right| \geqslant C\lambda^k \inf_{0 \leqslant j < k} \left|Df\left(f^j(x)\right)\right|.$$

Before continuing to the proof, we define first entry and first return maps. An open interval U is called *regularly returning* for f if $f^n(\partial U) \cap U = \emptyset$ for all $n \geqslant 0$. Define the integer-valued entry time $\epsilon(x)$ for x to U as $\epsilon(x) = \inf\{n \geqslant 0\colon\ f^n(x) \in U\}$ where the infimum exists.

Then the first entry map for U, ψ_U, is defined (where $\epsilon(x)$ is) by $\psi_U(x) = f^{\epsilon(x)}(x)$. The first return map to U, ϕ_U, is defined on U (where $\epsilon(f(x))$ is) by $\phi_U(x) = \psi_U(f(x))$.

2. Proofs

Let W' be a neighbourhood of $\mathrm{Crit}(f)$ such that $|Df_{|W'}(x)| < \frac{\delta}{4e}$ (where e is the base of the natural logarithm) for all $x \in W'$, let W be an open neighbourhood of $\mathrm{Crit}(f)$ compactly contained in W' and let $0 < \gamma = \mathrm{dist}(W, \partial W')$. Then

$$\left|\frac{D_{\delta,W'} f(x)}{Df(x)}\right| > \frac{\delta}{\delta/(4e)} = 4e \quad \text{for all } x \in W'. \tag{4}$$

We shall use the following Koebe Principle, proved in [2], in what follows:

Fact 2.1. *Koebe principle: Let I be a compact interval and $f: I \to I$ be a C^2 map with all critical points C^2 non-flat. Then there exists a continuous increasing function σ, $\sigma(0) = 0$, with the following property. If $J \subset T$ are open intervals and $n \in N$ is such that f^n is a diffeomorphism from T onto its image then, for every $x, y \in J$, we have*

$$\frac{Df^n(x)}{Df^n(y)} \geqslant \frac{e^{-\sigma(\max_{i=0}^{n-1}|f^i(T)|)\cdot\sum_{i=0}^{n-1}|f^i(J)|}}{(1+\nu(f^n(J), f^n(T)))^2},$$

where for open intervals $\bar{A} \subset B$, $\nu(A, B) = \frac{|A|}{\operatorname{dist}(A, \partial B)}$.

Fix positive $\eta < \operatorname{dist}(\partial W, \operatorname{Crit}(f))$ small enough such that $\sigma(\eta)|I| < 1$. By the Contraction Principle [4,1], given η there exists $\varepsilon > 0$ such that every open interval mapped by f^n, for some $n \geqslant 0$, to an interval (not in the immediate basin of a periodic attractor) of size less than or equal to ε is of size less than or equal to η.

Similarly there exists ε_0 such that every interval mapped by f^n, some $n \geqslant 0$, to an interval $\leqslant \varepsilon_0$ is of size $\leqslant \frac{1}{3}\varepsilon$ and smaller than γ.

Let V_c be a regularly returning neighbourhood of critical point c of size $\leqslant \varepsilon_0$. Let U be a connected component of the domain of the first entry map ψ_{V_c} to V_c, with $\psi_{V_c|U} = f^n$. Then $U, f(U), \ldots, f^n(U)$ are all of size $\leqslant \frac{1}{3}\varepsilon$ (since $|f^{n-i}(f^i(U))| \leqslant \varepsilon_0$).

Let $l < k \leqslant n$ be such that $f^l(U), \ldots, f^{k-1}(U)$ are all disjoint from W but $f^k(U) \cap W \neq \emptyset$. Since $k \leqslant n$, $|f^k(U)| \leqslant \frac{1}{3}\varepsilon$ and we can choose an interval T of size $\leqslant \varepsilon$ containing $f^k(U)$ such that $\nu(f^k(U), T) \leqslant 1$. Then T is also sufficiently small that there is an interval $T_0 \supset U$ with $f^k: T_0 \to T$ a diffeomorphism. This follows since $\max_{l \leqslant i \leqslant k} |f^i(T_0)| \leqslant \eta$ and $f^i(T_0) \supset f^i(U)$ and $f^i(U)$ is disjoint from W, so $f^i(T_0)$ does not contain any critical point.

We can then apply the Koebe Principle which gives distortion bound

$$\frac{Df^k(x)}{Df^k(y)} \geqslant \frac{e^{-\sigma(\eta)\cdot|I|}}{(1+\nu(f^k(U), T))^2} \geqslant \frac{1/e}{(1+1)^2} = \frac{1}{4e}$$

for all $x, y \in U$.

For intervals $A \subset B$ one says A is α-*well inside* B if $\operatorname{dist}(A, \partial B) \geqslant \alpha|A|$.

Proposition 1. *Given $\lambda > 1$, for each critical point c not in the basin of attraction of a periodic attractor, there exists a regularly returning interval V_* containing c such that the first return map ϕ_{V_*} is λ-expanding with respect to the δ-W-derivative.*

Proof. By the *real bounds* of van Strien and Vargas (Theorem A′ in [4]), there exist $\alpha > 0$ depending only on f and arbitrarily small V_c as above which are regularly returning and satisfy one of the following for the first return map ϕ_{V_c}:

(i) ϕ_{V_c} is defined on all of V_c and $|\phi_{V_c}(V_c)| \geqslant \alpha|V_c|$; we set $V_* = V_c$.
(ii) The connected component of the domain of definition of ϕ_{V_c} which contains c exists and is denoted V_*. V_* is α-well inside V_c.
(iii) The critical point is non-recurrent. Then c is accumulated on both sides by points of V_c not in the domain of ϕ_{V_c}. Any interval defined by a pair of such points is regularly returning. Let $V_* \ni c$ be such an interval which is α-well inside V_c.

In case (i) it follows immediately (since $\int_{V_*} |D\phi_{V_*}(x)|' dx \geqslant \alpha|V_*|$) that $\max_{x \in V_*} |D\phi_{V_*}(x)| > \alpha$.

Now we treat cases (ii), (iii). The first return map to V_*, ϕ_{V_*}, can be written at each x in its domain as a composition of $n = n(x)$ iterates of the first return map to V_c,

$$\phi_{V_*}(x) = \phi_{V_c}^n(x).$$

Let $U \subset V_*$ be the connected component of the domain of $\phi_{V_c}^n$ (note not of ϕ_{V_*}) containing x. Then $\phi_{V_c}^n(\partial U) \subset \partial V_c$ and $\phi_{V_c}^n(x) \in V_*$ so $\phi_{V_c}^n(U)$ contains a connected component of $V_c \setminus V_*$ and as such is of size $\geqslant \alpha|V_*|$, i.e.

$$\left|\phi_{V_c}^n(U)\right| \geqslant \alpha|V_*| \geqslant \alpha|U| \;\Rightarrow\; \max_{x \in U}\left|D\phi_{V_c}^n(x)\right| > \alpha.$$

Thus in each case we have a lower bound of α for the maximum of $D\phi^n_{V_c}$. Coming back to the general setting, $\phi^n_{V_c}$ restricted to U ($U = V_* = V_c$ in the renormalizable case (1) above) can be written as f^m for some m. Points in the orbit of U where $f^k(U) \cap W \neq \emptyset$ split up the orbit of U defined by f^m into N_W (say) sections outside of W. Each section outside of W contributes distortion of at most $4e$ from above. When $f^k(U) \cap W \neq \emptyset$ (and here we use inequality (4) and $|f^k(U)| < \gamma$ which implies $f^k(U) \subset W'$),

$$\frac{|D_{\delta,W'} f(f^k(x))|}{|Df(f^k(x)|} \geqslant 4e,$$

thus, setting $\delta_{V_c} = \max_{x\in V_c} |Df(x)|$, we have

$$\left|D_{\delta,W'}\phi^n_{V_c}(x)\right| \geqslant (4e)^{N_W-1}\frac{\delta}{\delta_{V_c}}\left|D\phi^n_{V_c}(x)\right|. \tag{5}$$

This, together with the distortion bound and $\max_{x\in U} |D\phi^n_{V_c}(x)| > \alpha$, implies

$$\min_{x\in U}\left|D_{\delta,W'}\phi^n_{V_c}(x)\right| \geqslant \alpha \cdot \frac{1}{(4e)^{N_W}} \cdot (4e)^{N_W-1}\frac{\delta}{\delta_{V_c}} = \frac{\alpha\delta}{4e}\frac{1}{\delta_{V_c}}. \tag{6}$$

This holds for all connected components of the domain of ϕ_{V_*} and so for all of the domain of ϕ_{V_*}. Constants α and δ are fixed, and choosing V_c sufficiently small $\frac{1}{\delta_{V_c}}$ is big, $\frac{\alpha\delta}{4e}\frac{1}{\delta_{V_c}} \geqslant \lambda$, and the proposition is proven. □

Recall that $\mathcal{A}$ is the union of immediate basins of periodic attractors, and that $\mathcal{P}$ is some small neighbourhood of the parabolic periodic points of f.

Fact 2.2 *(Mañé Hyperbolicity Theorem, [3,1] theorem III.5.1). Given* $\lambda > 1$ *and an open neighbourhood* V *of* $\mathrm{Crit}(f)$*, there exists* N_V *such that if* $x, f(x), \ldots, f^n(x) \in I \setminus (V \cup \mathcal{A} \cup \mathcal{P})$ *and* $n \geqslant N_V$ *then*

$$\left|Df^n(x)\right| \geqslant \lambda.$$

Proof. Proof of Main Theorem Given $\lambda > 1$ we fix regularly returning neighbourhoods $V_1, \ldots, V_p$ of the p (say) critical points such that the first return map to each of them is λ-expanding (by Proposition 1).

Let $V = \bigcup_{i=1}^p V_p$. Then by Fact 2.2 there exists N_V such that if $n \geqslant N_V$ then $|Df^n(x)| > \lambda$ for all x satisfying $x, f(x), \ldots, f^n(x) \in I \setminus (V \cup \mathcal{A} \cup \mathcal{P})$. Let $\nu = \inf\{|Df(x)|\colon x \in I \setminus (V \cup \mathcal{A}) > 0\}$.

Suppose we have an orbit $x, \ldots, f^n(x)$ in $I \setminus (\mathcal{A} \cup \mathcal{P})$. We set a coding a_j for this orbit where $a_j = 0$ if $f^j(x) \notin V$ and $a_j = i$ if $f^j(x) \in V_i$. There are $p+1$ characters $0, 1, \ldots, p$. In general for most j, a_j will equal 0.

Case 1: There are more than $C \cdot p + 1$ non-zero values a_j for some C. (We fix C at the end of this case.) This implies, according to the Pigeon-Hole Principle, the existence of an $i \in \{1, \ldots, p\}$ such that i appears at least $C + 1$ times, which corresponds to at least C first returns to V_i. In this part of the orbit there is therefore λ^C expansion.

Let us look to the left of the first occurrence of i in the coding. The non-zero terms are from a set of $p - 1$ characters $\{1, \ldots, p\} \setminus \{i\}$. Let i_1 be the rightmost such term. Then the sequence from the leftmost occurrence of i_1 to the rightmost corresponds to a composition of ($k \geqslant 0$) first returns to U_{i_1} and has thus 'expansion' $\geqslant \lambda^k \geqslant 1$.

Now to the left of the leftmost i_1, the non-zero terms come from a set $\{1, \ldots, p\} \setminus \{i, i_1\}$ of $p - 2$ characters. Continuing inductively, one can split up the sequence left of the first (leftmost) i into at most $p - 1$ sections corresponding to compositions of first return maps to critical neighbourhoods and at most p sections of the form $a00\cdots0$ (where $a \in \{1, \ldots, p\} \setminus \{i\}$ for all apart from possibly the leftmost section where $a = 0$ if $x \notin V$) separating the 'first return' sections. Each section $a00\cdots0$ can be at most ν^{N_V} contracting, since if there are more than $N_V - 1$ zeros we can apply Proposition 1 and get expansion of λ.

Arguing similarly to the right of the rightmost occurrence of i, we deduce for the whole sequence that

$$\left|D_{\delta,W} f^n(x)\right| \geqslant \left(\nu^{N_V}\right)^p \cdot \lambda^C \cdot \left(\nu^{N_V}\right)^p = \nu^{2N_V p} \cdot \lambda^C.$$

Since ν, N_V, λ and p are fixed constants, we can fix C sufficiently large that the expression on the right is greater than $\lambda > 1$, so $|D_{\delta,W} f^n(x)| \geqslant \lambda$.

Case 2: There are less than $C \cdot p + 1$ non-zero terms in the coding sequence. Suppose $n > N_V K(C \cdot p + 1)$ for some K, then there is a (maximal) block of at least $N_V K$ consecutive zeros in the sequence. Dividing the block into K parts and applying Fact 2.2, we get expansion of λ^K.

As before, to the left of this block one can bound contraction by $(\nu^{N_V})^p$ and likewise to the right. Thus for a sequence (orbit) of this type, we have

$$\left|D_{\delta,W} f^n(x)\right| > \left(\nu^{N_V}\right)^p \cdot \lambda^K \cdot \left(\nu^{N_V}\right)^p = \nu^{2N_V p} \cdot \lambda^K.$$

There exists K_0 such that, replacing K by K_0, the above expression is greater than λ, so $|D_{\delta,W} f^n(x)| \geqslant \lambda$.

Now let $N_0 = N_V K_0(C \cdot p + 1)$. If $n \geqslant N_0$ either Case 1 or Case 2 holds which proves the Main Theorem. $\square$

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