Abstract


Résumé


Let \((\Omega, \mathcal{A}, \mu, T)\) be a dynamical system where \(T\) is a bijective, bimeasurable and measure preserving map of \(\Omega\) onto \(\Omega\). By \(U\) we denote the operator on the space of all measurable functions on \(\Omega\) defined by \(Uf = f \circ T\), \((\mathcal{F}_i)_{i \in \mathbb{Z}}\) is a filtration, \(\mathcal{F}_i \subset T^{-1}\mathcal{F}_i = \mathcal{F}_{i+1}\). For a measurable function \(f\) we denote \(S_n(f) = \sum_{i=0}^{n-1} U^i f\). In [5], Maxwell and Woodroofe proved that if \(f \in L^2(\mathcal{F}_\infty) \ominus L^2(\mathcal{F}_{-\infty})\) is \(\mathcal{F}_0\)-measurable and

\[
\sum_{k=1}^{\infty} \frac{\|E(S_k(f) | \mathcal{F}_0)\|_2}{k^{3/2}} < \infty
\]

then there exists a martingale difference sequence \((U^i m)\) (adapted to the filtration \((\mathcal{F}_i)\)) approximating \((U^i f)\), i.e.

\[
\left\| E(S_k(f - m)) \right\|_2 = o(\sqrt{n}),
\]

which implies a central limit theorem for \((U^i f)\) (cf. [3]). In [6] Peligrad and Utev proved a new maximal inequality which implies that under (1) we get also the weak invariance principle. In [10] Volný found a method enabling to

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prove a nonadapted version of the Maxwell–Woodroofe’s CLT. In the article the martingale approximation (and hence a CLT) is proved for \( f \in L^2(F_\infty) \otimes L^2(F_{-\infty}) \) which satisfies
\[
\sum_{k=1}^{\infty} \frac{\|E(S_k(f) | F_0)\|_2}{k^{3/2}} < \infty, \quad \sum_{k=1}^{\infty} \frac{\|S_k(f) - E(S_k(f) | F_k)\|_2}{k^{3/2}} < \infty.
\] (3)

The idea of [10] is based on splitting of \( f \) into \( f = f' + f'' \) where \( f' = E(f | F_0) \) and applying an operator \( V \) which transforms the process \( (U^i f'') \) into an adapted sequence \( (U^i V f'') \). The assumption (3) then implies that both \( (U^i f') \) and \( (U^i V f'') \) satisfy (1). By the theorem of Maxwell and Woodroofe there exist martingale difference sequences \( (U^i m') \) and \( (U^i m'') \) adapted to \( (F_i) \) and approximating \( (U^i f') \) and \( (U^i f'') \) respectively. For \( m = m' + m'' \), \( (U^i m) \) is then a martingale difference sequence for which (2) holds true.

As shown in [4], the operator \( V \) need not correspond to any point mapping and the method thus does not give directly an invariance principle.

In this paper we will present a generalisation of the Peligrad–Utev’s maximal inequality to a larger class of processes, which will give a weak invariance principle for processes satisfying (3).

Let \( H \) be a subspace of \( L^2 \) for which \( UH \subset H \). To the operator \( U \) we associate a semigroup of contraction operators \( P_T k \), \( k = 1, 2, \ldots, \) (recall that \( Uf = f \circ T \)) on \( H \) which satisfies:

(i) \( P_T k = P_T k I \), \( k = 1, 2, \ldots; \)

(ii) \( P_T U = I \) where \( I \) is the identity operator;

(iii) if \( T f = 0 \) then \( (U^i f) \) is a martingale difference sequence;

we denote \( P_T 1 = P_T = P \).

**Proposition 1.** Let \( f \in H \) be such that
\[
\sum_{k=1}^{\infty} \frac{\| \sum_{i=1}^{k} P^i f \|_2}{k^{3/2}} < \infty.
\]
(4)

Then there exists a constant \( C \) such that for all \( n \geq 1 \),
\[
\left\| \max_{1 \leq k \leq n} \left\| \sum_{j=0}^{k-1} U^j f \right\|_2 \right\|_2 \leq C \sqrt{n} \left( \| f \|_2 + \sum_{k=1}^{n} \| \sum_{i=1}^{k} P^i f \|_2 \right).
\]
(5)

The proof is the same as the proof of Theorem 1 in [7]; in their case it can be taken \( H = L^2(F_0) \), \( P_T f = E(Uf | F_0) \), and \( U \) then replaced by \( U^{-1} \). The inequality holds also in \( L^p \) spaces with \( 1 \leq p < \infty \) (cf. [7]). In [8], Proposition 1, Tyran-Kamińska and Mackey presented the proof in an operator language and proved the inequality for \( P_T \) being the Perron–Frobenius operator. This way the inequality was proved for noninvertible endomorphisms (e.g. exact endomorphisms, where no nontrivial martingale difference sequence \( (U^i m) \) can exist). In the paper of Tyran-Kamińska and Mackey, \( T \) is a noninvertible endomorphism and the filtration is decreasing, given by \( G_i = T^{-i} A, i \geq 0 \). The endomorphism can, however be seen as a factor of an automorphism (cf. [2]); there thus exists a dynamical system \((\Omega_1, A_1, \mu_1, T_1)\) where \( T_1 \) is an automorphism, a filtration \((F_i) \) where \( F_i \subset T^{-1} F_i = F_{i+1} \), such that \((\Omega, A, \mu, T)\) is isomorphic to \((\Omega_1, F_0, \mu_1, T_1^{-1})\). We take \( H = L^2(F_0) \) and define \( P_T \) by \( P_T f = U E(f | F_{-1}) = E(Uf | F_0) \). The proposition above thus includes the case of Proposition 1 in [8].

**Theorem 1.** Let \( f \in L^2 \) be regular, i.e. \( F_{\infty} \)-measurable, \( E(f | F_{-\infty}) = 0 \). If
\[
\sum_{k=1}^{\infty} \frac{\|S_k(f) | F_0\|}{k^{3/2}} < \infty, \quad \sum_{k=1}^{\infty} \frac{\|S_k(f) - E(S_k(f) | F_k)\|}{k^{3/2}} < \infty,
\]
(6)

then the process of \( w_n(t) = (1/\sqrt{n}) \sum_{j=0}^{[nt]} U^j f \) weakly converges to the process \( \eta^2 W \) where \( W \) is the Brownian motion and \( \eta^2 \) is independent of \( W \).
Remark that if the measure μ is ergodic (i.e. for each A measurable, A = T⁻¹ A implies that A is either of measure 0 or of measure 1), n² is constant. In the nonergodic case we get n² constant on each ergodic component of μ (cf. [9]). In [8], a calculation of n² is given. For simplifying the notation we shall suppose that μ is ergodic.

For proving Theorem 1 we need to prove the central limit theorem for finite-dimensional distributions and the tightness (cf. [1]).

The central limit theorem for finite-dimensional distributions follows from (2) which has been proved in [10].

Let us define f' = E(f | F₀), f'' = f - f'. By the invariance principle of Peligrad and Utev (cf. [6]) we have the invariance principle for f'. It thus remains to prove the tightness for f''. It follows from the next proposition:

**Proposition 2.** Let f ∈ L² be F∞-measurable, E(f | F₀) = 0, and

\[ \sum_{k=1}^{∞} \frac{∥S_k(f) - E(S_k(f) | F_k)∥}{k^{3/2}} < ∞. \]

Then the process of \( w_n(t) = (1/\sqrt{n}) \sum_{j=0}^{[nt]} U^j f \) weakly converges to a Brownian motion.

**Proof.** Let \( F_i \) be a filtration with \( F_i \subset F_{i+1} = T^{-1} F_i \), \( P_{Tk}, k = 1, 2, \ldots \), a set of operators on \( H = L^2(F∞) \bigoplus L^2(F₀) \) defined by

\[ P_{Tk}h = U^{-k}h - E(U^{-k}h | F₀). \]

We have \( UH \subset H \) and we will prove that (i)–(iii) are fulfilled. Remark that

\[ U^k E(f | F_j) = E(U^k f | F_{j+k}). \]

(i) For \( k = 1 \) the statement is true by definition, suppose that it is true for \( k \).

\[ P_{Tk}^{k+1}h = P_T(U^{-k}h - E(U^{-k}h | F₀)) \]

\[ = U^{-1}(U^{-k}h - E(U^{-k}h | F₀)) - E(U^{-1}(U^{-k}h - E(U^{-k}h | F₀)) | F₀)) \]

\[ = U^{-(k+1)}h - E(U^{-(k+1)}h | F₀) = P_{Tk+1}h. \]

(ii) From \( h \in H \) it follows \( E(h | F₀) = 0 \) hence \( P_T Uh = h - E(h | F₀) = h. \)

(iii) We get 0 = \( U P_T h \) hence by (8), \( h = E(h | F₁) \), i.e. \( h \) is \( F₁ \)-measurable. We have \( h \in H \), hence \( E(h | F₀) = 0 \), therefore \( h \in L²(F₁) \bigoplus L²(F₀) \). Using (8) we get that \( U^k h \in L²(F_{k+1}) \bigoplus L²(F_k) \) hence \( (U^k h) \) is a martingale difference sequence.

From the fact that \( ∥S_k(f) - E(S_k(f) | F_k)∥₂ = ∥U^{-k}(S_k(f) - E(S_k(f) | F_k))∥₂ = ∥\sum_{j=1}^{k} P_{Tk}^j f ∥₂ \) we by Proposition 1 deduce the maximal inequality (5).

By [10] there is a martingale approximation (2) by a stationary martingale difference sequence and in the same way as in [6] or [8] we deduce the invariance principle. □

**References**