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Partial Differential Equations

Stable solutions of $-\Delta u = \mathrm{e}^u$ on $\mathbb{R}^N$

Alberto Farina

LAMFA, CNRS UMR 6140, université de Picardie Jules-Verne, faculté de mathématiques et d'informatique, 33, rue Saint-Leu, 80039 Amiens, France

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Abstract

In this Note we study C^2 solutions of the equation $-\Delta u = \mathrm{e}^u$ on the entire Euclidean space $\mathbb{R}^N$, with $N \geqslant 2$. We prove the non-existence of stable solutions for $N \leqslant 9$. In the two-dimensional case we also demonstrate a classification theorem for solutions which are stable outside a compact set. ***To cite this article: A. Farina, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

Solutions stables de $-\Delta u = \mathrm{e}^u$ dans $\mathbb{R}^N$. Cette Note porte sur l'étude des solutions de l'équation $-\Delta u = \mathrm{e}^u$ dans $\mathbb{R}^N$, $N \geqslant 2$. Nous démontrons la non-existence de solutions stables en dimension $N \leqslant 9$. En dimension $N = 2$, nous prouvons aussi un théorème de classification pour les solutions stables à l'extérieur d'un compact. ***Pour citer cet article : A. Farina, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

1. Introduction and main results

In this Note we study solutions of the semilinear partial differential equation:

$$-\Delta u = \mathrm{e}^u \quad \text{on } \mathbb{R}^N,\ N \geqslant 2. \tag{1}$$

The above problem arises in the theory of gravitational equilibrium of polytropic stars (see for instance [4,10,14] and the references therein). On the other hand, classification results for solutions defined on the entire Euclidean space are crucial to obtain a priori L^∞-bounds for solutions of semilinear boundary value problems in bounded domains (see for instance [2,7,12,13]).

Our main concern is to classify stable solutions of (1), or more generally, solutions of (1) which are stable (only) outside a compact set of $\mathbb{R}^N$. We recall that, given a domain $\Omega \subset \mathbb{R}^N$ (possibly unbounded), a solution $u \in C^2(\Omega)$ of $-\Delta u = \mathrm{e}^u$ is **stable** in Ω if:

$$\forall \psi \in C_c^1(\Omega) \quad Q_u(\psi) := \int_\Omega |\nabla \psi|^2 - \mathrm{e}^u \psi^2 \geqslant 0.$$

E-mail address: alberto.farina@u-picardie.fr.

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Inspired by the methods that we developed in our previous works [11,12] on the classification of solutions of the Lane–Emden equation on unbounded domains of $\mathbb{R}^N$, we are able to prove the following:

Theorem 1. *For $N \leqslant 9$, there is no stable C^2 solution of Eq.* (1).

Some remarks are in order.

Remarks 2. (i) Theorem 1 is sharp. Indeed, for every $N \geqslant 10$ Eq. (1) admits a radial stable solution. This follows from the analysis performed in [14], as was already remarked in [7].

(ii) The above theorem answers a question raised by H. Brezis [9].

(iii) For $N = 2$ and 3, and under the additional assumption that u is bounded above, the conclusion of the above Theorem 1 was previously obtained by E.N. Dancer [6]. The proof in [6] uses a completely different approach based on ideas originated with the work of L. Ambrosio and X. Cabré [1] in the study of a conjecture of E. De Giorgi [8]. We would like to point out that the assumption: u is bounded above, is crucial for this approach.

In [7], the author also proves that, for $N = 3$ Eq. (1) has no negative solution of finite Morse index. Here we focus on the two-dimensional case and prove a complete classification result for solutions which are stable outside a compact set of $\mathbb{R}^2$ (clearly this family of solutions includes all the solutions with finite Morse index, see for instance [6,11,12]). More precisely, we prove:

Theorem 3. *Let $u \in C^2(\mathbb{R}^2)$ be a solution of* (1) *with $N = 2$. Then, u is stable outside a compact set of $\mathbb{R}^2$ if and only if it is of the form*

$$u(x) = \ln\left[\frac{32\lambda^2}{(4+\lambda^2|x-x_0|^2)^2}\right], \quad \lambda > 0,\ x_0 \in \mathbb{R}^2. \tag{2}$$

Remark 4. The above Theorem 3 extends to distribution-solutions $u \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ such that $\mathrm{e}^u \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$. Indeed, the stability outside a compact set of $\mathbb{R}^2$, together with the local integrability of u, easily imply that $\int_{\mathbb{R}^2} \mathrm{e}^u < +\infty$. Therefore, a result of H. Brezis and F. Merle [3] yields that u is bounded above on the entire Euclidean plane and hence u is a classical solution of (1), by standard elliptic estimates. The result then follows by applying Theorem 3.

In view of the above results we are naturally led to the following:

Open Problem. *Let $N \geqslant 3$. Classify all the solutions of* (1) *which are stable outside a compact set of* $\mathbb{R}^N$.

2. Proofs

Theorem 1 is a consequence of the following:

Proposition 5. *Assume $N \geqslant 2$ and let Ω be a domain* (*possibly unbounded*) *of $\mathbb{R}^N$. Let $u \in C^2(\Omega)$ be a* **stable** *solution of*

$$-\Delta u = \mathrm{e}^u \quad \text{on } \Omega. \tag{3}$$

Then, for any integer $m \geqslant 5$ and any $\alpha \in (0,2)$ we have

$$\int_\Omega \mathrm{e}^{(2\alpha+1)u}\psi^{2m} \leqslant \left(\frac{m}{2-\alpha}\right)^{2\alpha+1} \int_\Omega \left(|\nabla\psi|^2 + |\psi||\Delta\psi|\right)^{2\alpha+1} \tag{4}$$

for all test functions $\psi \in C^2_c(\Omega)$ satisfying $0 \leqslant \psi \leqslant 1$ in Ω.

Proof. We split the proof into three steps.

Step 1. For any $\varphi \in C^2_c(\Omega)$ we have

$$\int_\Omega \left|\nabla\left(\mathrm{e}^{\alpha u}\right)\right|^2\varphi^2 = \frac{\alpha}{2}\int_\Omega \mathrm{e}^{(2\alpha+1)u}\varphi^2 + \frac{1}{4}\int_\Omega \mathrm{e}^{2\alpha u}\Delta\left(\varphi^2\right). \tag{5}$$

Multiply Eq. (3) by $e^{2\alpha u}\varphi^2$ and integrate by parts to find

$$\int_\Omega \nabla u \nabla\big(e^{2\alpha u}\big)\varphi^2 + \int_\Omega e^{2\alpha u}\nabla u \nabla\big(\varphi^2\big) = \int_\Omega e^{(2\alpha+1)u}\varphi^2,$$

and therefore

$$\int_\Omega e^{(2\alpha+1)u}\varphi^2 = \frac{2}{\alpha}\int_\Omega \big|\nabla\big(e^{\alpha u}\big)\big|^2\varphi^2 + \frac{1}{2\alpha}\int_\Omega \nabla\big(e^{2\alpha u}\big)\nabla\big(\varphi^2\big) = \frac{2}{\alpha}\int_\Omega \big|\nabla\big(e^{\alpha u}\big)\big|^2\varphi^2 - \frac{1}{2\alpha}\int_\Omega e^{2\alpha u}\Delta\big(\varphi^2\big).$$

The latter immediately implies identity (5).

Step 2. For any $\varphi \in C_c^2(\Omega)$ we have

$$\int_\Omega e^{(2\alpha+1)u}\varphi^2 \leqslant \frac{2}{2-\alpha}\int_\Omega e^{2\alpha u}\left[|\nabla\varphi|^2 - \frac{\Delta(\varphi^2)}{4}\right]. \tag{6}$$

Inserting the function $\psi = e^{\alpha u}\varphi$ in the quadratic form Q_u we get

$$\begin{aligned}\int_\Omega e^{(2\alpha+1)u}\varphi^2 &\leqslant \int_\Omega \big|\nabla\big(e^{\alpha u}\big)\big|^2\varphi^2 + \int_\Omega e^{2\alpha u}|\nabla\varphi|^2 + \frac{1}{2}\int_\Omega \nabla\big(e^{2\alpha u}\big)\nabla\big(\varphi^2\big)\\ &= \int_\Omega \big|\nabla\big(e^{\alpha u}\big)\big|^2\varphi^2 + \int_\Omega e^{2\alpha u}|\nabla\varphi|^2 - \frac{1}{2}\int_\Omega e^{2\alpha u}\Delta\big(\varphi^2\big).\end{aligned} \tag{7}$$

Using (5) in the latter inequality we obtain

$$\int_\Omega e^{(2\alpha+1)u}\varphi^2 \leqslant \frac{\alpha}{2}\int_\Omega e^{(2\alpha+1)u}\varphi^2 + \frac{1}{4}\int_\Omega e^{2\alpha u}\Delta(\varphi^2) + \int_\Omega e^{2\alpha u}|\nabla\varphi|^2 - \frac{1}{2}\int_\Omega e^{2\alpha u}\Delta\big(\varphi^2\big),$$

which gives the desired conclusion.

Step 3. End of the proof. For any $\psi \in C_c^2(\Omega)$ satisfying $0 \leqslant \psi \leqslant 1$ in Ω we set $\varphi = \psi^m$. Inserting φ in (6) we obtain

$$\int_\Omega e^{(2\alpha+1)u}\psi^{2m} \leqslant \frac{m}{2-\alpha}\int_\Omega e^{2\alpha u}\psi^{2(m-1)}\big[|\nabla\psi|^2 - \psi\Delta\psi\big]$$

and an application of Hölder's inequality leads to

$$\int_\Omega e^{(2\alpha+1)u}\psi^{2m} \leqslant \frac{m}{2-\alpha}\left(\int_\Omega \big[e^{2\alpha u}\psi^{2(m-1)}\big]^{\frac{2\alpha+1}{2\alpha}}\right)^{\frac{2\alpha}{2\alpha+1}}\left(\int_\Omega \big[|\nabla\psi|^2 + |\psi||\Delta\psi|\big]^{2\alpha+1}\right)^{\frac{1}{2\alpha+1}}.$$

Now, we observe that $m \geqslant 5$ implies $(m-1)\frac{(2\alpha+1)}{\alpha} \geqslant 2m$ and thus $\psi^{(m-1)\frac{(2\alpha+1)}{\alpha}} \leqslant \psi^{2m}$ in Ω, since $0 \leqslant \psi \leqslant 1$ everywhere in Ω. Therefore,

$$\int_\Omega e^{(2\alpha+1)u}\psi^{2m} \leqslant \frac{m}{2-\alpha}\left(\int_\Omega e^{(2\alpha+1)u}\psi^{2m}\right)^{\frac{2\alpha}{2\alpha+1}}\left(\int_\Omega \big[|\nabla\psi|^2 + |\psi||\Delta\psi|\big]^{2\alpha+1}\right)^{\frac{1}{2\alpha+1}},$$

which proves the claim. □

Proof of Theorem 1. Suppose to the contrary that Eq. (1) admits a stable solution for $N \leqslant 9$. Fix an integer $m \geqslant 5$ and choose $\alpha \in (0,2)$ such that $N - 2(2\alpha+1) < 0$ (notice that this is always possible since $N \leqslant 9$). For every $R > 0$ and every $x \in \mathbb{R}^N$, consider the function $\phi_R(x) = \phi(\frac{|x|}{R})$, where $\phi \in C_c^2(\mathbb{R})$ satisfies $0 \leqslant \phi \leqslant 1$ everywhere on $\mathbb{R}$ and

$$\phi(t) = \begin{cases} 1 & \text{if } |t| \leqslant 1,\\ 0 & \text{if } |t| \geqslant 2.\end{cases} \tag{8}$$

Now we apply Proposition 5 with $\Omega = \mathbb{R}^N$ and $\psi = \phi_R$ to get

$$\forall R > 0 \quad \int\limits_{|x|<R} e^{(2\alpha+1)u} \leqslant C R^{N-2(2\alpha+1)}$$

where C is a positive constant independent on R. Letting $R \to +\infty$ in the latter inequality we obtain $\int_{\mathbb{R}^N} e^{(2\alpha+1)u} = 0$, a contradiction. This concludes the proof. □

Proof of Theorem 3. Since u is stable outside a compact set of $\mathbb{R}^2$ there is $R_0 > 0$ such that Proposition 5 holds true with $\Omega = \mathbb{R}^2 \setminus \overline{B(0,R)}$, where $B(0,R_0)$ denotes the ball centered at the origin and of radius R_0. For every $R > R_0 + 3$ and every $x \in \mathbb{R}^2$, consider the function $\psi_R \in C_c^2(\mathbb{R}^2 \setminus \overline{B(0,R_0)})$ satisfying

$$\psi_R(x) = \begin{cases} \xi & \text{if } |x| \leqslant R_0 + 3, \\ \phi_R & \text{if } |x| \geqslant R_0 + 3, \end{cases} \tag{9}$$

where ϕ_R was defined in the proof of Theorem 1 and ξ is any function belonging to $C^2(\mathbb{R}^2)$ and such that $0 \leqslant \xi \leqslant 1$ on $\mathbb{R}^2, \xi = 0$ in the ball centered at the origin and of radius $R_0 + 1$ and $\xi = 1$ outside the ball centered at the origin and of radius $R_0 + 2$. Since $Q_u(\psi_R) \geqslant 0$ we get $\int_{\mathbb{R}^2} e^u < +\infty$ and hence u must be of the form (2) by a well-known result of W. Chen and C. Li [5]. Conversely, any function given by (2) is stable outside a large ball of $\mathbb{R}^2$. Clearly, it is enough to prove the claim for $x_0 = 0$ and $\lambda > 0$. To this end, we observe that there exists $R = R(\lambda) > 1$ such that $e^{u(x)} \leqslant \frac{1}{4|x|^2 \ln^2(|x|)}$ for $|x| > R$, and that, $\forall \psi \in C_c^1(\mathbb{R}^2 \setminus \overline{B(0,R)})$ we have $\int_{|x|>R} |\nabla\psi|^2 - \frac{\psi^2}{4|x|^2 \ln^2(|x|)} \geqslant 0$ (the latter follows immediately from the fact that $\ln^{\frac{1}{2}}(|x|)$ is a positive solution of $-\Delta u = \frac{1}{4|x|^2 \ln^2(|x|)} u$ outside the closed unit ball of $\mathbb{R}^2$). Combining these two properties we obtain the desired conclusion. □

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