



Mathematical Physics

Local existence of classical solutions for the Einstein–Euler system using weighted Sobolev spaces of fractional order

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Abstract

We prove the existence of a class of local in time solutions, including static solutions, of the Einstein–Euler system. This result is the relativistic generalisation of a similar result for the Euler–Poisson system obtained by Gamblin (1993). As in his case the initial data of the density do not have compact support but fall off at infinity in an appropriate manner. An essential tool in our approach is the construction and use of weighted Sobolev spaces of fractional order. Moreover, these new spaces allow us to improve the regularity conditions for the solutions of evolution equations. The details of this construction, the properties of these spaces and results on elliptic and hyperbolic equations will be presented in a forthcoming article. **To cite this article:** *U. Brauer, L. Karp, C. R. Acad. Sci. Paris, Ser. I 345 (2007).*

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Résumé

Existence locale de solutions classiques pour le système Einstein–Euler en utilisant des espaces de Sobolev avec poids à ordre fractionnaire. Nous prouvons l'existence d'une classe de solutions locales en temps, incluant des solutions statiques, du système d'Einstein–Euler. Notre résultat est la généralisation relativiste d'un résultat similaire pour le système d'Euler–Poisson obtenu par Gamblin (1993). Comme dans son cas, les données initiales de la densité ne sont pas à support compact mais décroissent à l'infini d'une façon appropriée. L'un des outils essentiels dans notre approche est la construction et l'usage des espaces de Sobolev à poids et d'ordre fractionnaire. De plus, ces nouveaux espaces nous permettent d'améliorer les conditions de régularité pour les solutions des équations d'évolution. Les détails de cette construction, les propriétés de ces espaces et quelques résultats sur des équations elliptiques et hyperboliques seront présentés dans un futur article. **Pour citer cet article :** *U. Brauer, L. Karp, C. R. Acad. Sci. Paris, Ser. I 345 (2007).*

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Version française abrégée

Dans cette Note on va étudier le problème de Cauchy pour le système de Euler–Einstein pour une densité décroissant à l’infini d’une façon convenable. L’un des outils principaux sera l’usage des espaces de Sobolev à poids et d’ordre fractionnaire. Les équations d’Euler sont écrites comme un système symétrique hyperbolique de la forme

$$\begin{pmatrix} g^2 u^\nu & w P^\nu_\beta \\ w P^\nu_\alpha & \Gamma_{\alpha\beta} u^\nu \end{pmatrix} \nabla_\nu \begin{pmatrix} w \\ u^\beta \end{pmatrix} = 0, \quad (1)$$

où $\Gamma_{\alpha\beta} = g_{\alpha\beta} + 2u_\alpha u_\beta$ est définie positive. Remarquons que, en sus des équations d’évolution précédentes, nous avons une équation de contrainte pour la vitesse u^α c’est-à-dire $g_{\alpha\beta} u^\alpha u^\beta = -1$. La variable $w = M(\epsilon)$ est une fonction non linéaire de la densité.

$$w = M(\epsilon) = \int_0^\epsilon \frac{1}{g} \frac{1}{\tilde{\epsilon} + p} \sqrt{f'(\tilde{\epsilon})} d\tilde{\epsilon} = \epsilon^{(\gamma-1)/2}, \quad g = \frac{\epsilon}{(\epsilon + p)\bar{K}}, \quad \bar{K} = \frac{\gamma - 1}{2\sqrt{K}\gamma} \quad (2)$$

et elle doit être introduite pour régulariser le système pour $\rho = 0$. Les équations d’Einstein consistent en deux ensembles d’équations, celles d’évolution et celles de contrainte. En utilisant la condition coordonnée harmonique $g^{\alpha\beta} g^{\gamma\delta} (\partial_\gamma g_{\beta\delta} - \frac{1}{2} \partial_\delta g_{\beta\gamma}) = 0$, les premières prennent la forme

$$\begin{aligned} \partial_t g_{\alpha\beta} &= h_{\alpha\beta 0}, \\ g^{ab} \partial_t h_{\gamma\delta a} &= g^{ab} \partial_a h_{\gamma\delta 0}, \\ -g^{00} \partial_t h_{\gamma\delta 0} &= 2g^{0a} \partial_a h_{\gamma\delta 0} + g^{ab} \partial_a h_{\gamma\delta b} + C_{\gamma\delta\alpha\beta\rho\sigma}^{\epsilon\zeta\eta\kappa\lambda\mu} h_{\epsilon\zeta\eta} h_{\kappa\lambda\mu} g^{\alpha\beta} g^{\rho\sigma} - 16\pi T_{\gamma\delta} + 8\pi g^{\rho\sigma} T_{\rho\sigma} g_{\gamma\delta}, \end{aligned} \quad (3)$$

et ainsi elles peuvent être écrites ensemble avec (1) comme un système symétrique hyperbolique. Les équations de contrainte, après avoir été traitées par la méthode conforme, constituent un système d’équations couplées de la forme

$$\bar{\Delta}\phi - \frac{1}{8}\bar{R}\phi + \frac{1}{8}(\bar{A}_{ab}\bar{A}^{ab})\phi^{-7} = -2\phi^{-3}\bar{y}, \quad (4)$$

$$(\Delta_L W)^b = \bar{v}^b, \quad (5)$$

où $(\Delta_L W)^b = (\bar{\Delta} W)^b + \frac{1}{3}\bar{D}^b(\bar{D}_a W^a) + \bar{R}^b_a W^a$, et $\bar{\Delta}$ dénote ici l’opérateur de Laplace–Beltrami par rapport à la métrique \bar{h} . Les nouveaux espaces que nous utilisons ici ont été introduits par Triebel et sont une généralisation des espaces à poids de Sobolev d’ordre entier, lesquels possèdent pour δ réel et k non négatif la norme suivante $(\|u\|_{k,\delta}^*)^2 = \sum_{|\alpha| \leq k} \int |\langle x \rangle^{\delta+|\alpha|} \partial^\alpha u|^2 dx$, où $\langle x \rangle = 1 + |x|$. Cependant nous cherchons une norme équivalente qui soit plus appropriée à notre dessein. Soient $\{\psi_j\}$ une partition dyadique de l’unité. Les espaces fonctionnels sont alors donnés par la Définition 1.1. Ces espaces possèdent quelques propriétés que nous avons résumées dans le Lemme 1.2.

Notre résultat principal est énoncé dans le Théorème 2.1. Les solutions obtenues par ce théorème-ci incluent des solutions du système Euler–Einstein qui sont statiques et de symétrie sphérique. Elles sont donc une généralisation relativiste d’un résultat obtenu par Gamblin [3] pour le système d’Euler–Poisson ainsi qu’une généralisation d’un résultat de Rendall [7] concernant le système d’Euler–Einstein relativiste.

La démonstration comporte trois parties : d’abord nous résolvons les équations de contraintes elliptiques en appliquant la méthode conforme dans nos espaces. Puis on construit des données initiales pour les équations du fluide : à partir des données initiales pour les équations de contrainte, nous construisons les données initiales pour les équations d’Euler au moyen du Théorème 2.2. Finalement on établit l’existence locale de la solution de l’équation d’évolution symétrique hyperbolique énoncée dans le Théorème 2.3. Notons que, pour l’équation d’Einstein dans le vide, la première étape n’est pas nécessaire.

1. Introduction

We consider the Einstein–Euler system describing a relativistic self-gravitating perfect fluid. The unknowns in the equations are functions of t and x^a , where x^a ($a = 1, 2, 3$) are Cartesian coordinates of \mathbb{R}^3 . The alternative notation $x^0 = t$ will also be used and Greek indices will take the values 0, 1, 2, 3 in the following. The evolution of the gravitational field is described by the Einstein equations $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$ where $G_{\alpha\beta}$ is the Einstein tensor of the

spacetime metric $g_{\alpha\beta}$ and $T_{\alpha\beta}$ is the energy-momentum tensor of the matter. In the case of a perfect fluid the latter takes the form $T^{\alpha\beta} = (\epsilon + p)u^\alpha u^\beta + pg^{\alpha\beta}$ where ϵ is the energy density, p is the pressure and u^α is the four-velocity. The quantity u^α is required to satisfy the normalisation condition $g_{\alpha\beta}u^\alpha u^\beta = -1$. The Euler equations describing the evolution of the fluid take the form $\nabla_\alpha T^{\alpha\beta} = 0$. To obtain a determined system of equations it is necessary to specify a relation between ϵ and p (equation of state). The choice we make here is one which has been used for astrophysical problems. It is an analogue of the well known polytropic equation of state of the non-relativistic theory given by: $p = f(\epsilon) = K\epsilon^\gamma$, where $K, \gamma \in \mathbb{R}^+, 1 < \gamma$. In this setting Rendall [7] proved a local in time existence theorem for initial data with compact support for the density ϵ . Rendall however worked with C^∞ data and did therefore, restrict the equation of state.

The initial value problem for the Einstein–Euler system will be treated by writing the equations as a symmetric hyperbolic system in harmonic coordinates, which is given by $g^{\alpha\beta}g^{\gamma\delta}(\partial_\gamma g_{\beta\delta} - \frac{1}{2}\partial_\delta g_{\beta\gamma}) = 0$. When this condition is imposed the Einstein equations imply a system of quasilinear wave equations. To get a symmetric hyperbolic system these are reduced to first order by introducing auxiliary variables $h_{\alpha\beta\gamma} = \partial_\gamma g_{\alpha\beta}$. They can then be written in the following form

$$\begin{aligned} \partial_t g_{\alpha\beta} &= h_{\alpha\beta 0}, \\ g^{ab}\partial_t h_{\gamma\delta a} &= g^{ab}\partial_a h_{\gamma\delta 0}, \\ -g^{00}\partial_t h_{\gamma\delta 0} &= 2g^{0a}\partial_a h_{\gamma\delta 0} + g^{ab}\partial_a h_{\gamma\delta b} + C_{\gamma\delta\alpha\beta\rho\sigma}^{\epsilon\zeta\eta\kappa\lambda\mu} h_{\epsilon\zeta\eta} h_{\kappa\lambda\mu} g^{\alpha\beta} g^{\rho\sigma} - 16\pi T_{\gamma\delta} + 8\pi g^{\rho\sigma} T_{\rho\sigma} g_{\gamma\delta}. \end{aligned} \tag{6}$$

The Euler system $\nabla_\alpha T^{\alpha\beta} = 0$, is written in a symmetric hyperbolic form by decomposing it into two orthogonal components; the first is given by $u_\beta \nabla_\nu T^{\nu\beta} = 0$ and the second one given by $P_{\alpha\beta} \nabla_\nu T^{\nu\beta} = 0$ (with $P_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta$). Moreover, since we are dealing with a situation in which the density vanishes (at infinity) we have to regularise the system by introducing a new matter variable

$$w = M(\epsilon) = \int_0^\epsilon \frac{1}{g\tilde{\epsilon} + p} \sqrt{f'(\tilde{\epsilon})} d\tilde{\epsilon} = \epsilon^{(\gamma-1)/2}, \quad g = \frac{\epsilon}{(\epsilon + p)\bar{K}}, \quad \bar{K} = \frac{\gamma - 1}{2\sqrt{K\gamma}}. \tag{7}$$

These steps result in a system of the form

$$\begin{pmatrix} g^2 u^\nu & w P^\nu_\beta \\ w P^\nu_\alpha & \Gamma_{\alpha\beta} u^\nu \end{pmatrix} \nabla_\nu \begin{pmatrix} w \\ u^\beta \end{pmatrix} = 0, \tag{8}$$

where $\Gamma_{\alpha\beta} = g_{\alpha\beta} + 2u_\alpha u_\beta$ is positive definite. Note that we have, besides the above evolution equations, a constraint equation for the velocity u^α , namely $g_{\alpha\beta}u^\alpha u^\beta = -1$. The evolution equations (6) and (8) form a uniform symmetric hyperbolic system, the Einstein–Euler system. Recall that a (uniform) symmetric hyperbolic system is a system of differential equations of the form $L[U] = \sum_{\alpha=0}^3 A^\alpha(U; x, t)\partial_\alpha U + B(U; x, t) = 0$ with symmetric matrices A^α and for which the matrix A^0 is uniformly positive definite.

For the system introduced above we want to consider a initial value problem for which the initial density falls off at infinity in an appropriate way. The way the matter variables the new matter variable (7) and the pressure appear in the Euler and in the Einstein equation provide us with the following complication. The Einstein evolution equations (6) contain the term $T^{\alpha\beta} = (\epsilon + p)u^\alpha u^\beta + pg^{\alpha\beta}$, which is a function of ϵ, p and u^α . The pressure p and the energy density ϵ are connected via the equation of state $p = f(\epsilon) = K\epsilon^\gamma$. Therefore we have to estimate ϵ and p , in the corresponding norm of our function spaces, by w , which is an algebraic function of ϵ as given by Eq. (7). This estimate results in an algebraic relation between the order of the functional space k and the coefficient γ of the equation of state $1 < \gamma \leq \frac{2+k}{k}$. We do not want to restrict γ but instead interpret this inequality as an restriction on k and since we want also to improve the regularity conditions for the solutions of the Einstein–Euler system, we are naturally lead to consider weighted Sobolev spaces of fractional order. Moreover, using such spaces we can improve the regularity of the solutions.

The weighted Sobolev spaces of integer order below were introduced by Cantor and independently by Nirenberg and Walker [6]. For real δ and nonnegative integer k we define the

$$(\|u\|_{k,\delta}^*)^2 = \sum_{|\alpha| \leq k} \int |\langle x \rangle^{\delta+|\alpha|} \partial^\alpha u|^2 dx, \tag{9}$$

where $\langle x \rangle = 1 + |x|$. The space $H_{s,k}$ is the completion of $C_0^\infty(\mathbb{R}^3)$ under the norm (9). There are several ways to extend these spaces to fractional order. Originally Triebel [8] extended these spaces by using a definition which involves a double integral. His definition is a natural generalisation of the norm (9). However, the double integral causes many difficulties as one turns to prove certain properties (algebra, embedding act.) of the space $H_{s,\delta}$. Therefore we are looking for an equivalent norm. Let $K_j = \{x: 2^{j-3} \leq |x| \leq 2^{j+2}\}$ ($j = 1, 2, \dots$) and $K_0 = \{x: |x| \leq 4\}$. Let $\{\psi_j\}_{j=0}^\infty$ be a sequence of $C_0^\infty(\mathbb{R}^3)$ such that $\psi_j(x) = 1$ on K_j , $\text{supp}(\psi_j) \subset \bigcup_{l=j-4}^{j+3} K_l$, for $j \geq 1$, $\text{supp}(\psi_0) \subset K_0 \cup K_1$ and $|\partial^\alpha \psi_j(x)| \leq C_\alpha 2^{-|\alpha|j}$. We denote by H^s the Bessel potential spaces with the norm ($p = 2$)

$$\|u\|_{H^s}^2 = c \int (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi,$$

where \hat{u} is the Fourier transform of u . Also, for a function f , $f_\varepsilon(x) = f(\varepsilon x)$.

Definition 1.1 (Weighted fractional Sobolev spaces: infinite sum of semi norms). For $s \geq 0$ and $-\infty < \delta < \infty$,

$$(\|u\|_{H_{s,\delta}})^2 = \sum_j 2^{(3/2+\delta)2j} \|(\psi_j u)_{(2^j)}\|_{H^s}^2. \quad (10)$$

The space $H_{s,\delta}$ is the set of all temperate distributions with a finite norm given by (10).

The spaces $H_{s,\delta}$ defined by the norm (10) are equivalent to Triebel's definition incorporating double integrals and coincide with Cantor–Nirenberg–Walker space (9) when s is a nonnegative integer [8].

Lemma 1.2 (Properties of the fractional $H_{s,\delta}$ spaces). The spaces $H_{s,\delta}$ have the following properties:

1. (Algebra) For $s_1, s_2 \geq s$, $s_1 + s_2 > s + \frac{3}{2}$ and $\delta_1 + \delta_2 \geq \delta - \frac{3}{2}$, $\|uv\|_{H_{s,\delta}} \leq C \|u\|_{H_{s_1,\delta_1}} \|v\|_{H_{s_2,\delta_2}}$.
2. (Compact embedding) For $s' < s$ and $\delta' < \delta$ the embedding $H_{s,\delta} \hookrightarrow H_{s',\delta'}$ is compact.
3. (Moser's type estimates) Let $F: \mathbb{R}^1 \rightarrow \mathbb{R}^1$ be C^∞ such that $F(0) = 0$. Then $\|F(u)\|_{H_{s,\delta}} \leq C(\|u\|_{L^\infty}) \|u\|_{H_{s,\delta}}$.
4. For $1 \leq \gamma$, $s < \gamma + \frac{1}{2}$ and $u \geq 0$, $\|u^\gamma\|_{H_{s,\delta}} \leq C(\|u\|_{L^\infty}) \|u\|_{H_{s,\delta}}$. This inequality has been proven for the H^s spaces by Kateb [4].

2. The principal result

Theorem 2.1 (Main result).

1. **Solution of the constrains equations (14) and (15):** Let $2 \leq s$ and $-\frac{3}{2} < \delta < -\frac{1}{2}$. Given free initial data $\bar{h}_{ab} - \delta_{ab} \in H_{s,\delta}(\mathbb{R}^3)$, $\bar{A}_*^{ab} \in H_{s-1,\delta+1}(\mathbb{R}^3)$, $\bar{y}(\epsilon) \in H_{s-2,\delta+2}(\mathbb{R}^3)$, $v^b(u^\alpha) \in H_{s-2,\delta+2}(\mathbb{R}^3)$. Then there exists a unique solution ϕ , K_{ab} of the constraint equations (14) and (15) such that $\phi - 1 \in H_{s,\delta}(\mathbb{R}^3)$, $K_{ab} \in H_{s-1,\delta+1}(\mathbb{R}^3)$.
2. **Solution of the evolution equations (6) and (8):** Let $s, \delta \in \mathbb{R}$, $\frac{7}{2} < s < \frac{2}{\gamma-1} + \frac{3}{2}$, and $-\frac{3}{2} < \delta < -\frac{1}{2}$. Given the solutions of the constraints (14) and (15), assume moreover that $\bar{y}(\epsilon) \in H_{s-1,\delta+2}(\mathbb{R}^3)$, $v^b(u^\alpha) \in H_{s-1,\delta+2}(\mathbb{R}^3)$. Then there exists a $T > 0$ and a unique solution $U = (w, u^0 - 1, u^\alpha, g_{\alpha\beta})$ of the Einstein–Euler system, with $g_{\alpha\beta} - \eta_{\alpha\beta} \in C^0([0, T], H_{s,\delta}(\mathbb{R}^3)) \cap C^1([0, T], H_{s-1,\delta}(\mathbb{R}^3))$, $w, u^0 - 1, u^\alpha \in C^0([0, T], H_{s-1,\delta+2}(\mathbb{R}^3)) \cap C^1([0, T], H_{s-2,\delta+2}(\mathbb{R}^3))$.

The proof consists of three parts: First we solve the elliptic constraints applying the established methods introduced by Cantor and others (see [1] and references within), and others for our spaces. For details we refer to our forthcoming paper. The next step concerns the construction of the initial data for the fluid equations: Starting with the initial data for the constrain equations, we construct the initial data for the Euler equations (8) by means of Theorem 2.2. The last step finally refers to the local existence of the symmetric hyperbolic evolution equation given by an appropriate generalisation of the well known existence and uniqueness theorem for symmetric–hyperbolic systems. To write down the constraint equations it is convenient to introduce the second fundamental form of the initial surface, which is given by $-\frac{1}{2}\partial_t g_{ab} = K_{ab}$. Let n^α denote the unit normal to the hypersurface, $\delta_\beta^\alpha + n^\alpha n_\beta$ be the projection on it and

define $\bar{u}^\alpha = (\delta_\alpha^\beta + n^\alpha n^\beta)u^\beta$, $z = T_{\alpha\beta}n^\alpha n^\beta$, $j^\alpha = (\delta_\gamma^\alpha + n^\alpha n_\gamma)T^{\gamma\beta}n_\beta$. This leads to $z = \epsilon(1 + g_{ab}\bar{u}^a\bar{u}^b) + pg_{ab}\bar{u}^a\bar{u}^b$, $j^a = (\epsilon + p)\bar{u}^a(1 + g_{bc}\bar{u}^b\bar{u}^c)^{1/2}$. If R_{ab} denotes the Ricci tensor of the induced metric on the initial hypersurface, $R = g^{ab}R_{ab}$ is its scalar curvature and $(^3)\nabla$ its associated covariant derivative, then Einstein constraint equations read:

$$R - K_{ab}K^{ab} + (g^{ab}K_{ab})^2 = 16\pi z, \tag{11}$$

$$(^3)\nabla_b K^{ab} - (^3)\nabla^b (g^{bc}K_{bc}) = -8\pi j^a. \tag{12}$$

We turn now to the conformal method which allows us to construct the solutions of the constraint equations (11) and (12).

As we will outline elsewhere [2] instead of constructing w, u^a from z, j^a it turns out that is more useful to introduce some auxiliary quantities. So we start with $w = \epsilon^{(\gamma-1)/2}$ and $y = z^{(\gamma-1)/2}$. Now we consider the following map $(w, u^a) \rightarrow (y, \frac{j^a}{z})$ which is given by (13). Theorem 2.2 below shows that in fact C^∞ and a local diffeomorphism if $\frac{\|j\|}{z} < 1$.

Theorem 2.2 (Reconstruction theorem for the initial data). *Let Φ be the mapping from \mathbb{R}^4 to \mathbb{R}^4 defined by*

$$\Phi(w, \bar{u}^a) = \left(w[(1 + g_{ab}\bar{u}^a\bar{u}^b) + Kw^2g_{ab}\bar{u}^a\bar{u}^b]^{(\gamma-1)/2}, \frac{(1 + Kw^2)(1 + g_{bc}\bar{u}^b\bar{u}^c)^{1/2}\bar{u}^a}{(1 + g_{bc}\bar{u}^b\bar{u}^c) + Kw^2g_{bc}\bar{u}^a\bar{u}^b} \right). \tag{13}$$

Then under the dominate energy condition the map Φ is an C^∞ diffeomorphism from a closed half-space of \mathbb{R}^4 onto the region $G = \{(y, x^a) : 0 \leq y, \delta_{ab}x^ax^b < 1\}$. Here K is the constant of the state equation $p = K\epsilon^\gamma$.

For the proof we refer to [2]. Once the matter data for the constraints equations are constructed, the standard conformal method will be applied. We will just briefly outline this procedure with the necessary modifications we have to perform. Parts of the data (the so-called free data) are chosen, and the constraints imply four elliptic equations for the remaining parts. The free initial data are: $(\bar{h}_{ab}, \bar{A}_*^{ab}, \bar{y}, \bar{v}^b)$, where $v^b = \frac{j^b}{z}$ and y is given by $w = \epsilon^{\frac{\gamma-1}{2}}$ and $y = z^{\frac{\gamma-1}{2}}$. Moreover we have performed a conformal transformation of the metric: $h_{ab} = \phi^4\bar{h}_{ab}$. We assume the maximal slice condition for K_{ab} , that is $h^{ab}K_{ab} = 0$. Let \bar{A}_*^{ab} be any smooth symmetric tensor which has zero trace with respect to \bar{h}_{ab} . We are looking for solutions using $\bar{A}^{ab} = \bar{A}_*^{ab} + \bar{D}^aW^b + \bar{D}^bW^a - \frac{2}{3}\bar{h}^{ab}\bar{D}_kW^k$ and $K^{ab} = \phi^{-10}\bar{A}^{ab}$. Furthermore we have $y = \phi^{4(\gamma-1)}\bar{y}$ $v^a = \phi^{-2}\bar{v}^a$. The transformed constraints are to be solved for the scalar function ϕ and the vector field W^b

$$\bar{\Delta}\phi - \frac{1}{8}\bar{R}\phi + \frac{1}{8}(\bar{A}_{ab}\bar{A}^{ab})\phi^{-7} = -2\phi^{-3}\bar{y}, \tag{14}$$

$$(\Delta_L W)^b = \bar{v}^b, \tag{15}$$

where $(\Delta_L W)^b = (\bar{\Delta}W)^b + \frac{1}{3}\bar{D}^b(\bar{D}_aW^a) + \bar{R}_a^bW^a$, here $\bar{\Delta}$ denotes the Laplace–Beltrami operator with respect to the metric \bar{h} . The solution (ϕ, W^b) to Eqs. (14) and (15) is obtained by the development of elliptic theory in $H_{s,\delta}$ spaces. Then the full initial data can be obtained by inverting the above process. In order to obtain such a solution (ϕ, W^b) , we proceed as follows: First, for given $\bar{y} \in H_{s-2,\delta+2}$, we know by the extended Katab result as given in Lemma 1.2 that $\bar{z} \in H_{s-2,\delta+2}$, moreover by assumption we have $\bar{v}^b \in H_{s-2,\delta+2}$ and by the multiplication property 1.2, we conclude that $j^b \in H_{s-2,\delta+2}$. Therefore the momentum constraint (15) is solved for W^b which results in $W^b \in H_{s,\delta}$. Secondly using W^b and the free initial data A_*^{ab} , A^{ab} is constructed using the composition mentioned above. Finally with z and A^{ab} given the Lichnerowicz equation (14) is solved for ϕ . The resulting theorems are equivalent to the statements as in (1) of Theorem 2.1.

Remark 1. During our work we found out that D. Maxwell [5] studied the vacuum Einstein constraint equations using fractional weighted Sobolev spaces. He obtained solutions for the constrain equations under the condition $\frac{3}{2} < s$, improving earlier result obtained by Bartnik [1]. We recall that our principal motivation is to adjust the regularity of the solution to the Einstein–Euler system (6) and (8) for each parameter γ of the state equation $p = K\epsilon^\gamma$.

A principal tool for dealing with the evolution equations (6) and (8) is a existence theorem which is a generalisation of the well known existence theorem for the H^s spaces.

Theorem 2.3 (Local existence for quasilinear symmetric–hyperbolic systems). Let $A^0, A^k \in C^\infty(\mathbb{R}^3 \times \mathbb{R}; \mathbb{R}^{l \times l})$, $B \in C^\infty(\mathbb{R}^3 \times \mathbb{R}; \mathbb{R}^{l \times l})$ be coefficients which define the quasilinear symmetric–hyperbolic system $A_{\alpha\beta}^0(U; x, t) \frac{\partial U^\beta}{\partial t} + \sum_{k=1}^3 A_{\alpha\beta}^k(U, x, t) \frac{\partial U^\beta}{\partial x^k} + B_{\alpha\beta}(U; x, t) U^\beta = 0$. Let $U(x, 0) \in H_{s,\delta}(\mathbb{R}^3)$ and let the initial conditions be chosen such that the condition $C \delta_{\alpha\beta} U^\alpha U^\beta \leq A_{\alpha\beta}^0 U^\alpha U^\beta \leq C^{-1} \delta_{\alpha\beta} U^\alpha U^\beta$, $C \in \mathbb{R}^+$, is satisfied. Let $\frac{5}{2} < s$ and $-\frac{3}{2} \leq \delta$. Then there exists a $T > 0$ which depends on the $H_{s,\delta}$ norm of the initial data and there exists a unique solution $U(x, t) \in C^0([0, T], H_{s,\delta}) \cap C^1([0, T], H_{s-1,\delta})$.

Again for a proof of this theorem we refer to [2]. Our principal result is therefore proven by applying Theorem 2.2, together with the initial data of the gravitational field, as solutions of the constraints, and finally using the theorem mentioned above. Note that in our main result we have demanded a bound from above on the differentiability on the initial data namely $\frac{7}{2} < s < \frac{2}{\gamma-1} + \frac{3}{2}$. The reason for this is that w which appears in the evolution equations (8) is a function of ϵ as given by $w = \epsilon^{(\gamma-1)/2}$.

References

- [1] R. Bartnik, Phase space for the Einstein equations, *Comm. Anal. Geom.* 13 (5) (2005) 845–885.
- [2] U. Brauer, L. Karp, Weighted Sobolev spaces of fractional order: properties and applications to elliptic, hyperbolic and Einstein’s equations, in preparation.
- [3] P. Gamblin, Solution régulière á temps petit pour l’équation d’Euler–Poisson, *Comm. Partial Differential Equations* 18 (5 & 6) (1993) 731–745.
- [4] D. Kateb, On the boundedness of the mapping $f \mapsto |f|^\mu$, $\mu > 1$ on Besov spaces, *Math. Nachr.* 248 (249) (2003) 110–128.
- [5] D. Maxwell, Rough solutions of the Einstein constraint equations, *J. Reine Angew. Math.* 590 (2006) 1–29.
- [6] L. Nirenberg, H. Walker, The null spaces of elliptic differential operators in \mathbb{R}^n , *J. Math. Anal. Appl.* 42 (1973) 271–301.
- [7] A.D. Rendall, The initial value problem for a class of general relativistic fluid bodies, *J. Math. Phys.* 33 (2) (1992) 1047–1053.
- [8] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, Johann Ambrosius Barth, 1995.