Differential Geometry

A uniqueness result for maximal surfaces in Minkowski 3-space

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Received 5 July 2006; accepted after revision 12 May 2007
Available online 20 June 2007
Presented by Thierry Aubin

Abstract

In this Note, we study the Dirichlet problem associated to the maximal surface equation. We prove the uniqueness of bounded solutions to this problem in unbounded domains in $\mathbb{R}^2$. To cite this article: L. Mazet, C. R. Acad. Sci. Paris, Ser. I 344 (2007). © 2007 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

Résumé


Version française abrégée

On considère l’espace de Minkowski de dimension 3 $\mathbb{L}^3$ c’est-à-dire $\mathbb{R}^3$ muni de la métrique Lorentzienne $\langle x, y \rangle_\mathbb{L} = x_1y_1 + x_2y_2 - x_3y_3$. Dans $\mathbb{L}^3$, les surfaces de type espace ayant une courbure moyenne nulle sont appelées surfaces maximales. Ces surfaces sont localement des graphes : $x_3 = v(x_1, x_2)$ ; la surface étant maximale la fonction $v$ satisfait l’équation aux dérivées partielles ($\ast$). Ainsi l’étude des surfaces maximales passent par la résolution du problème de Dirichlet associé à cette équation.

Dans cette note, nous démontrons l’unicité des solutions bornées. Plus précisément, si $\Omega$ est un domaine non borné de $\mathbb{R}^2$ et $v$ et $v'$ sont deux solutions bornées de ($\ast$) sur $\Omega$ telles que $v = v'$ sur $\partial \Omega$, alors $v = v'$ sur $\Omega$ (Theorem 2.2). Remarquons que dans le cas des domaines bornés, cette unicité est une conséquence du principe du maximum. Notons que le résultat ci-dessus a été annoncé par A.A. Klyachin dans [5] mais aucune preuve ne semble avoir été écrite depuis.

Un des intérêts du résultat est qu’il permet de montrer l’unicité pour les solutions de l’équation des surfaces minimales de $\mathbb{R}^3$ ($\ast\ast$) (Corollaire 2.4) ; en effet on dispose d’une correspondance entre surfaces maximales de $\mathbb{L}^3$ et surfaces minimales de $\mathbb{R}^3$.

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doi:10.1016/j.crma.2007.05.014
La démonstration du résultat d’unicité repose sur des idées utilisées par P. Collin et R. Krust dans [2]. Toutefois, la grosse différence est que pour appliquer ces techniques, nous avons besoin d’une majoration du gradient des solutions de (*). La Section 2 présente la preuve du résultat d’unicité sous réserve de l’estimé de gradient. L’idée est que si on a deux solutions distinctes \( v \) et \( v’ \), on est capable de minorer le maximum de \( |v - v'| \) par un minorant qui ne peut rester borné car le domaine \( \Omega \) n’est pas compact.

Dans la Section 3, nous démontrons la majoration du gradient (Lemme 2.3). En fait, nous montrons qu’en un point où le gradient \( \nabla v \) est grand (c’est-à-dire proche de 1 en norme), la différence \( v - v’ \) est obligatoirement faible. Ainsi le maximum de \( |v - v'| \) est forcément atteint en un point où le gradient est faible. La démonstration de ce résultat repose sur la correspondance entre surfaces maximales et surfaces minimales qui permet d’utiliser des résultats développés dans [8] pour les suites divergentes de solutions de l’équation des surfaces minimales (**) en \( \Omega \).

1. Introduction

We consider the Minkowski space–time \( \mathbb{L}^3 \) i.e. \( \mathbb{R}^3 \) with the following Lorentzian metric \( \langle x, y \rangle_L = x_1 y_1 + x_2 y_2 - x_3 y_3 \). We define \( |x|_L^2 = \langle x, x \rangle \). A vector is said to be spacelike if \( |x|_L^2 > 0 \) and a surface \( S \) of class \( C^1 \) is said to be spacelike if \( | \cdot |_L^2 \) is positive definite on the tangent space to \( S \). Such a surface is locally the graph of a function over a domain in \( \mathbb{R}^2 \).

If \( v \) is a function in a domain \( \Omega \) in \( \mathbb{R}^2 \), the graph of \( v \) is spacelike if and only if \( |\nabla v| < 1 \). The function \( v \) is then Lipschitz continuous and it extends to the closure \( \overline{\Omega} \). In the paper, we assume that \( \partial \Omega \) is sufficiently regular for such an extension to exist: for example, \( \Omega = \overline{\Omega} - \{ \text{points} \} \) with \( \partial \Omega \) smooth. We denote by \( \phi \) the trace \( v|_{\partial \Omega} \) of \( v \) on the boundary. The maximal area problem in the class of spacelike surfaces consists in solving the following variational problem:

\[
\max_v \int_\Omega 1 - |\nabla v|^2 \, dx, \quad v|_{\partial \Omega} = \phi.
\]

The critical points of this functional are the solutions of the maximal surface equation:

\[
\text{div} \frac{\nabla v}{\sqrt{1 - |\nabla v|^2}} = 0.
\]  

(*)

The maximal area problem is then linked to the Dirichlet problem associated to (*): to find a solution \( v \) of (*) in \( \Omega \) such that \( v|_{\partial \Omega} = \phi \). This Dirichlet problem has been already studied by several authors, for example see [1] and [6].

In this Note, we prove the uniqueness of bounded solutions to the Dirichlet problem. More precisely, if \( \Omega \) is an unbounded domain and \( \phi \) is a bounded continuous function on \( \partial \Omega \), we prove that, if it exists, a bounded solution \( v \) of (*) in \( \Omega \) with \( v|_{\partial \Omega} = \phi \) is unique (Theorem 2.2). The study of the uniqueness is important in the construction of certain moduli spaces of maximal surfaces (see [3] and [4]).

In fact our result has already been stated by A.A. Klyachin in [5]. In his note, Klyachin states several results about the Dirichlet problem for (*) in unbounded domains and any dimension. But he does not give any proof. He uses the notion of capacity: if \( \Omega \) is a bounded domain and \( P, Q \subset \Omega \) satisfy \( P \cap Q = \emptyset \), the capacity of \( (P, Q; \Omega) \) is:

\[
\text{cap}(P, Q; \Omega) = \inf \int_\Omega |\nabla u|^2
\]

where the inf is taken on all Lipschitz continuous functions \( u \) on \( \Omega \) with \( u = 0 \) on \( P \) and \( u = 1 \) on \( Q \). If \( \Omega \) is unbounded \( \Omega \) is said to be parabolic if for any compact subset \( P \subset \Omega : \lim_{R \to +\infty} \text{cap}(P, \partial B_R \cap \Omega; B_R \cap \Omega) = 0 \) where \( B_R \) denotes the centered ball of radius \( R \). One of the results stated by Klyachin is the following (Theorem 3 in [5])

**Theorem.** Let \( \Omega \subset \mathbb{R}^n \) be an unbounded parabolic domain. Let \( v_1 \) and \( v_2 \) be two bounded solutions of (*) in \( \Omega \). We define \( \phi_1 = v_1|_{\partial \Omega} \) and \( \phi_2 = v_2|_{\partial \Omega} \). Then: \( v_1(x) \leq v_2(x) + \sup_{\partial \Omega} (\phi_1 - \phi_2) \).

Since every unbounded domain in \( \mathbb{R}^2 \) is parabolic, this theorem implies our uniqueness result.
This uniqueness result for the maximal surface equation is also important for the study of the Dirichlet problem associated to the minimal surface equation. The graph of a function $v$ over a domain $\Omega \subset \mathbb{R}^2$ is a surface in $\mathbb{R}^3$ with its standard Euclidean metric. It has vanishing mean curvature if $u$ satisfies the following partial differential equation:

$$\text{div} \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} = 0.$$  (**) 

This equation implies that there exists locally a function $v$ such that:

$$dv = d\Psi_u = \frac{u_x}{\sqrt{1 + |\nabla u|^2}} \, dy - \frac{u_y}{\sqrt{1 + |\nabla u|^2}} \, dx$$

(here $u_x$ and $u_y$ are the first derivatives of $u$). $v = \Psi_u$ is called the conjugate function to $u$ and a simple computation shows that $v$ is a solution of $(\ast)$. Then the uniqueness for solutions of $(\ast)$ implies uniqueness for solutions of (**)).

The proof of our uniqueness result uses the same idea as P. Collin and R. Krust in [2]. However, to apply this idea, we need to prove an estimate for the first derivatives of $v$ in a subdomain of $\Omega$; this is done in Lemma 2.3.

2. The uniqueness result

Let $\Omega \subset \mathbb{R}^2$ be a domain and $v$ a solution of the maximal surface equation $(\ast)$. In the following, the quantity $\sqrt{1 - |v|^2}$ will be denoted by $w$. We define the 1-form $\alpha_v$ as follows:

$$\alpha_v = \frac{v_x}{w} \, dy - \frac{v_y}{w} \, dx$$

where $v_x$ and $v_y$ are the first derivatives of $v$. The maximal surface equation is then equivalent to $d\alpha_v = 0$.

First, we need a technical lemma:

**Lemma 2.1.** Let $v$ and $v'$ be two functions. Let $P$ be a point in $\Omega$ and $\varepsilon > 0$ such that $|v|(P) \leq 1 - \varepsilon$ and $|v'(P)| \leq 1 - \varepsilon$. There exists a constant $C(\varepsilon)$ that depends only on $\varepsilon$ such that, at the point $P$, we have:

$$\left( (\nabla v - \nabla v') \cdot \left( \frac{\nabla v}{w} - \frac{\nabla v'}{w'} \right) \right) \geq C(\varepsilon) \left| \nabla v \frac{w'}{w} - \nabla v' \right|.$$  (1) 

Let us notice that in Collin–Krust’s paper [2], there is the same lemma, but they do not need any bound on the gradient to get the estimate (1).

**Proof.** Let $w$ denote $w_v$ and $w'$ denote $w_{v'}$. We define $n = (-v_x, -v_y, 1)/w$ and $n' = (-v'_x, -v'_y, 1)/w'$. We have $|n|^2 = 1$ and $|n'|^2 = 1$, so: $(\nabla v - \nabla v') \cdot (\nabla v - \nabla v') = (w' - wn), (n' - n) = (w + w')(1 - |n, n'|) = \frac{w + w'}{2}|(n' - n)|^2$. Since $|\nabla v| \leq 1 - \varepsilon$ and $|\nabla v'| \leq 1 - \varepsilon$, there exists $C_1(\varepsilon) > 0$ such that

$$(w + w')/2 \geq C_1(\varepsilon).$$  (2) 

Moreover:

$$\frac{|(n' - n)|^2}{|\nabla v/w - \nabla v'/w'|^2} = \frac{1}{\frac{1}{w} - \frac{1}{w'}} - \left( \frac{1}{w} - \frac{1}{w'} \right)^2.$$ 

Let $x \in \mathbb{R}^2$ be $\nabla v/w$ and $x'$ be $\nabla v'/w'$. Thus $1/w = \sqrt{1 + |x|^2}$ and $1/w' = \sqrt{1 + |x'|^2}$. Since $\nabla v$ and $\nabla v'$ are bounded by $1 - \varepsilon$, there exists $R(\varepsilon)$ such that $|x|$ and $|x'|$ are bounded by $R(\varepsilon)$. Hence:

$$\frac{|(n' - n)|^2}{|\nabla v/w - \nabla v'/w'|^2} = 1 - \frac{1/w - 1/w'}{2} = 1 - \frac{(1 + |x|^2 - 1 + |x'|^2)^2}{|x - x'|^2} = 1 - \frac{|x - x'|^2}{|x - x'|^2} \left( \frac{|x| + |x'|}{\sqrt{1 + |x|^2} + \sqrt{1 + |x'|^2}} \right)^2 > 0.$$
By continuity and since $|x|$ and $|x'|$ are bounded by $R(\varepsilon)$, there exists a constant $C_2(\varepsilon) > 0$ such that:

$$1 - \frac{|x| + |x'|}{\sqrt{1 + |x|^2 + \sqrt{1 + |x'|^2}}}^2 > C_2(\varepsilon).$$

(3)

Then in combining (2) and (3), we get (1) with $C(\varepsilon) = C_1(\varepsilon)C_2(\varepsilon)$. □

We denote by $d$ the usual distance in $\mathbb{R}^2$ and by $d_\Omega$ the intrinsic metric in $\Omega$ i.e. $d_\Omega(p, q)$ is the infimum of the length of all paths in $\Omega$ going from $p$ to $q$. Let $\delta > 0$, we denote by $\Omega_\delta$ the set $\{p \in \Omega \mid d_\Omega(p, \partial \Omega) > \delta\}$. We then can write our uniqueness result:

**Theorem 2.2.** Let $\Omega$ be an unbounded domain in $\mathbb{R}^2$ and $\phi$ a bounded continuous function on $\partial \Omega$. Let $v$ and $v'$ be two bounded solutions of $(*)$ in $\Omega$ with $v|_{\partial \Omega} = \phi = v'|_{\partial \Omega}$. Then $v = v'$.

**Proof.** Let $v$ and $v'$ be two such solutions. We assume that $\sup v - v' > 0$ and we denote this supremum by $4\delta$. Let $\alpha \in [2\delta, 3\delta]$ be chosen such that $\tilde{\Omega} = \{v > v' + \alpha\}$ has smooth boundary. Since $2\delta \leq \alpha \leq 3\delta$ and $v$ and $v'$ are 1-Lipschitz continuous, $\tilde{\Omega} \subset \Omega_\delta$. We have the following lemma:

**Lemma 2.3.** There exists $\varepsilon > 0$ such that, in $\tilde{\Omega}$, $|\nabla v| \leq 1 - \varepsilon$ and $|\nabla v'| \leq 1 - \varepsilon$.

Before proving this lemma, we finish the proof of Theorem 2.2. Let $\tilde{v}$ denote $v - v' - \alpha$ and $\tilde{\alpha}$ denote $\alpha_v - \alpha_{v'}$.

For $r > 0$, we define $\bar{\Omega}_r = \{p \in \tilde{\Omega} \mid |p| < r\}$ and $C_r = \{p \in \tilde{\Omega} \mid |p| = r\}$. Since $\tilde{v} = 0$ on $\partial \bar{\Omega}_r \setminus C_r$ and $\tilde{\alpha}$ is closed, we have: $\int_{C_r} \tilde{v} \tilde{\alpha} = \int_{\bar{\Omega}_r} \tilde{v} \tilde{\alpha} = \int_{\partial \bar{\Omega}_r} \tilde{v} \tilde{\alpha}$. Since $\tilde{d} \tilde{\alpha} \wedge \tilde{\alpha} = ((\nabla \tilde{v} - \nabla \tilde{v}') \cdot (\nabla v - \nabla v')) dx \wedge dy$, Lemmas 2.1 and 2.3 imply that: $C(\varepsilon) \int_{\partial \bar{\Omega}_r} |\tilde{\alpha}|^2 \leq \int_{C_r} \tilde{v} \tilde{\alpha}$.

Let $r_0$ be such that $\mu = C(\varepsilon) \int_{\partial \bar{\Omega}_{r_0}} |\tilde{\alpha}|^2 > 0$. In $\tilde{\Omega}$, $\tilde{v}$ is bounded by $2\delta$, so:

$$\mu + C(\varepsilon) \int_{\bar{\Omega}_{r_0}} |\tilde{\alpha}|^2 \leq 2\delta \int_{C_r} |\tilde{\alpha}|.$$

Let us denote $\int_{C_r} |\tilde{\alpha}|$ by $\eta(r)$. By Schwartz’s Lemma: $\eta^2(r) \leq \ell(C_r) \int_{C_r} |\tilde{\alpha}|^2 \leq 2\pi r \int_{C_r} |\tilde{\alpha}|^2$. Hence $\int_{C_r} |\tilde{\alpha}|^2 \geq \frac{\eta^2(r)}{2\pi r}$ and $\int_{r_0}^{r} \frac{\eta^2(t)}{2\pi t} \leq \int_{\partial \bar{\Omega}_{r_0}} |\tilde{\alpha}|^2$. Finally:

$$\mu + C(\varepsilon) \int_{r_0}^{r} \frac{\eta^2(t)}{2\pi t} \leq 2\delta \eta(r).$$

(4)

Let $y$ be the solution of the following Cauchy problem: $y'(t) = C(\varepsilon) \frac{\eta^2(t)}{4\pi t}, y(r_0) = \frac{\mu}{4\delta}$. The function $y$ is defined on $[r_0, r_1]$ with $r_1 = r_0 \exp(\frac{16\delta^3}{\mu C(\varepsilon)})$ and satisfies: $\frac{4\delta}{\mu} - 1 \frac{y(t)}{y(t)} = C(\varepsilon) \ln \frac{r_1}{r_0}$. By (4), $\eta(t) \geq y(t)$ on $[r_0, r_1)$ and, since $\lim_{t \to r_1} y(t) = +\infty$, we get a contradiction, indeed $\eta$ is bounded. Then $v = v'$. □

As said in the introduction, Theorem 2.2 has a consequence for solution of the minimal surface equation:

**Corollary 2.4.** Let $\Omega$ be an unbounded simply-connected domain in $\mathbb{R}^2$. Let $u$ and $u'$ be two solutions of $(**)$ in $\Omega$ such that $\Psi_u$ and $\Psi_{u'}$ are bounded in $\Omega$ and $\Psi_u = \Psi_{u'}$ on $\partial \Omega$. Then $u = u'$ is constant.

We need the simple-connectedness hypothesis to ensure that $\Psi_u$ and $\Psi_{u'}$ are well defined.

**Proof.** $\Psi_u$ and $\Psi_{u'}$ are two solutions of $(*)$ in $\Omega$, so, by Theorem 2.2, $\Psi_u = \Psi_{u'}$. Hence $\nabla u = \nabla u'$ and $u - u'$ is constant. □

To end Theorem 2.2 proof, we have to prove Lemma 2.3.
3. The gradient estimate

This section is devoted to the proof of the gradient estimate in Lemma 2.3; this is the last step in Theorem 2.2 proof.

Proof of Lemma 2.3. If Lemma 2.3 is not true, we can assume that sup \( D \tilde{\Omega} \) \( |\nabla v| = 1 \). Thus there exists \( (p_n) \) a sequence in \( \tilde{\Omega} \) such that \( |\nabla v|(p_n) \to 1 \). Let \( O \) be the point \((0,0)\). Let \( r_n \) be the affine rotation of \( \mathbb{R}^2 \) such that \( r_n(O) = p_n \) and \( R_n^{-1}(\nabla v(p_n)) = (|\nabla v|(p_n),0) \) \((R_n \) is the linear rotation associated to \( r_n)\). We define \( v_n = v \circ r_n \) which is a solution of \((*)\) in \( \Omega_n = r_n^{-1}\Omega \). We have \( \nabla v_n = R_n^{-1}\nabla v \), so \( \nabla v_n(O) \to (1,0) \). In the same way, we define \( v'_n = v' \circ r_n \).

Let \( I(a,b) \subset \mathbb{R}^2 \) be the segment \([a,b]\times\{0\}\) \((a < b)\). Let \( \varepsilon \) be positive, \( \varepsilon \) will be fixed later but let us notice that \( \varepsilon/\delta \) will be small. Let \( D(a,b) \) denote the set \( \{p \in \mathbb{R}^2 | d(p,I(a,b)) < \varepsilon\} \), \( D(a,b) \) is the union of a rectangle of width \( 2\varepsilon \) and length \( b - a \) and two half-disks of radius \( \varepsilon \) (see Fig. 1).

For every \( n \), we define \( a_n \) and \( b_n \) by: \( a_n = \inf\{a \leq 0 | D(a,0) \subset \Omega_n\} \) and \( b_n = \sup\{b \geq 0 | D(0,b) \subset \Omega_n\} \). Since \( \varepsilon < \delta \) and \( O \in (\Omega_n)_\delta = r_n^{-1}(\Omega_\delta) \) (because \( p_n \in \Omega_\delta \)), \( b_n > 0 \) and \( a_n < 0 \); moreover \( D(a_n,b_n) \subset \Omega_n \). We define \( b_\infty = \lim \inf b_n \), \( b_\infty > 0 \), \( b_\infty \) may take the value \(+\infty\); by taking a subsequence, we assume that \( b_\infty = \lim b_n \). Then we define \( a_\infty = \lim \sup a_n \), \( a_\infty < 0 \), \( a_\infty \) may take the value \( -\infty \); as above we can assume that \( a_\infty = \lim a_n \). Let \( \beta = \min(\varepsilon/2,|a_\infty|,b_\infty) \), let \( A \) denote \( a_\infty + \beta \) if \( a_\infty > -\infty \) and any negative number if not and \( B \) denote \( b_\infty - \beta \) if \( b_\infty < +\infty \) and any positive number if not. For \( n \) big enough, \( D(A,B) \subset \Omega_n \) (see Fig. 1).

Since \( D(A,B) \) is simply connected, for each large \( n \) in \( \mathbb{N} \), there exists \( u_n \) a function on \( D(A,B) \) such that \( du_n = \alpha_{v_n} \). Besides the function \( u_n \) satisfies the minimal surface equation \((**)*\). The graph of \( u_n \) is a minimal surface in \( \mathbb{R}^3 \) with the Euclidean metric. Since \( du_n = \alpha_{v_n} \), we have \( dv_n = -d\Psi_{u_n} \), then \( v_n \) is the opposite of the conjugate function to \( u_n \). Since \( \nabla v_n(O) \to (1,0) \), \( \nabla u_n(O) \to +\infty \) and \( \nabla v_n(O) \to (0,1) \). Then \( \{y=0\} \cap D(A,B) \) is a divergence line for the sequence \( (u_n) \) (see [7,8]). This implies that if \( A - \varepsilon < s < t < B + \varepsilon \):

\[
\lim v_n(t,s) - v_n(s,0) = t - s.
\]

By hypothesis, \( v \) is bounded by some \( M > 0 \) so \( v_n \) is bounded by \( M \). This implies that \( A \) and \( B \) are bounded thus \( a_\infty \) and \( b_\infty \) cannot take infinite value; indeed \( (5) \) implies \( B - A \leq 2M \). Hence \( A = a_\infty + \beta \) and \( B = b_\infty - \beta \). By the definition of \( b_\infty \), the point \((b_\infty,0)\) which is in \( D(a_\infty + \beta, b_\infty - \beta) \) is at a distance less than \( 2\varepsilon \) from \( \partial \Omega_n \) for large \( n \) (see Fig. 1). So there exists, for each large \( n \), a point \( q_n \) in \( \partial \Omega_n \) such that \( d_{\Omega_n}(q_n,(b_\infty,0)) \leq 2\varepsilon \). By \( (5) \), we can assume that for \( n \) large enough: \( v_n(b_\infty,0) - v_n(O) \geq b_\infty - \varepsilon \); thus:

\[
v_n(O) = v_n(O) - v_n(b_\infty,0) + v_n(b_\infty,0) \leq \varepsilon - b_\infty + v_n(b_\infty,0).
\]

We recall that \( \phi \) is the boundary value of \( v \) and \( v' \). Moreover \( v'_n(O) \geq \phi(q_n) - d_{\Omega_n}(O,q_n) \geq \phi(q_n) - 2\varepsilon - b_\infty \). So \( v_n(O) - v'_n(O) \leq 5\varepsilon \). Since the sequence \( (p_n) \) is in \( \tilde{\Omega} \), \( v(p_n) - v'(p_n) > a \) and \( v_n(O) - v'_n(O) > a \). Hence, if \( \varepsilon \) is chosen such that \( \varepsilon < a/5 \), we get a contradiction and Lemma 2.3 is proved. \( \square \)

References


