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Probability Theory

An invariance principle for non-adapted processes

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Abstract

We present an invariance principle for a non-adapted stationary sequence of random variables, conditional with respect to the σ-algebra of invariant sets. It is a generalization of an invariance principle of Wu and Woodroofe (2004, Corollary 3) using a method introduced by Volný (2006). An example shows that the method cannot be used directly for a generalization of the invariance principle of Peligrad and Utev (2005). ***To cite this article: J. Klicnarová, D. Volný, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Résumé

Principe d'invariance pour processus non-adaptés. Nous présentons un principe d'invariance conditionnel (par rapport à la tribu des ensembles invariants) pour une suite stationnaire non-adaptée de variables aléatoires. Il généralise le principe d'invariance de Wu et Woodroofe (2004, Corollary 3) en utilisant la méthode introduite par Volný (2006). A l'aide d'un exemple, nous montrons que la méthode ne donne pas une généralisation du principe d'invariance de Peligrad et Utev (2005). ***Pour citer cet article : J. Klicnarová, D. Volný, C. R. Acad. Sci. Paris, Ser. I 345 (2007).***

Version française abrégée

Soit $(\Omega, \mathcal{A}, P)$ un espace probabilisé muni d'une transformation $T : \Omega \to \Omega$ bijective et bimeasurable, préservant la probabilité P, et soit $(\mathcal{F}_k, k \in \mathbb{Z})$ une filtration telle que $\mathcal{F}_k = T^{-k}\mathcal{F}_0$ et $\mathcal{F}_k \subset \mathcal{F}_{k+1}$.

Dans [9, Corollary 3], Wu et Woodroofe ont démontré que si

$$\left\| E(S_n|\mathcal{F}_0)\right\|_2 = \mathrm{o}\left(\frac{\sqrt{n}}{\log^q n}\right)$$

avec $S_n = \sum_{i=1}^n X_i$, où $(X_i = f \circ T^i, i \in \mathbb{N})$, $f \in L^p$, $p > 2$, $q \geqslant 2$, et si f est $\mathcal{F}_0$-mesurable, alors le principe d'invariance faible a lieu.

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L'objet de cette Note est de donner une version non-adaptée du principe d'invariance de Wu et Woodroofe [9, Corollary 3]. Pour une filtration $(\mathcal{F}_k, k \in \mathbb{Z})$ donnée, nous démontrons le théorème pour des variables régulières, c'est à dire, dans $L^2(\mathcal{F}_\infty) \ominus L^2(\mathcal{F}_{-\infty})$ et utilisons la technique développée dans [7].

Wu et Woodroofe ont présenté leur principe d'invariance sous une forme conditionnelle pour des chaînes de Markov stationnaires. Le cas non-adapté est étudié pour des processus stationnaires généraux et sous des conditions correspondant à celles d'un théorème limite conditionnel, nous démontrons que le théorème a lieu pour presque toutes composantes ergodiques de la mesure invariante (la probabilité pour laquelle le processus est stationnaire).

La méthode permet de trouver des versions non-adaptées des théorèmes limites démontrés à l'aide d'une approximation martingale dans un espace L^2. Elle utilise une isometrie de $L^2(\mathcal{F}_\infty) \ominus L^2(\mathcal{F}_{-\infty})$ qui rend les suites de $X_k \in L^2(\mathcal{F}_\infty) \ominus L^2(\mathcal{F}_k)$ adaptées. Cette isometrie, en général, ne correspond à aucune application ponctuelle et par conséquent pour démontrer le principe d'invariance de Peligrad et Utev [5] d'autres idées sont nécessaires.

1. Introduction and results

Let $(\Omega, \mathcal{A}, P)$ be a probability space and $T : \Omega \to \Omega$ be a bijective bimeasurable and measure preserving transformation. By $\mathcal{I}$ we denote the σ-field of invariant sets from $\mathcal{A}$, i.e. of $A \in \mathcal{A}$ such that $T^{-1}A = A$. Recall that a T-invariant probability measure P is called ergodic if for all $A \in \mathcal{I}$ it is $P(A) = 0$ or $P(A) = 1$. Without loss of generality (cf., e.g., [6]) we can suppose that $\mathcal{A}$ is a Borel σ-algebra of a Polish space and therefore there exist regular conditional probabilities P_ω with respect to $\mathcal{I}$ which are T-invariant and ergodic probability measures (ergodic components of P).

Let f be a measurable function; then the sequence $(X_i = f \circ T^i, i \in \mathbb{N})$ is strictly stationary. Let us define

$$S_n = \sum_{i=1}^n X_i.$$

By a filtration we mean a sequence of σ-fields $(\mathcal{F}_k, k \in \mathbb{Z})$ such that $\mathcal{F}_k = T^{-k}\mathcal{F}_0$ and $\mathcal{F}_k \subset \mathcal{F}_{k+1}$.

Let $(S_n(t))_n$ be a sequence of random variables with values in $D[0,1]$ defined by

$$S_n(t) := \frac{1}{\sqrt{n}} S_{\lfloor nt \rfloor}.$$

We say that $S_n(t)$ converge in law to Ψ conditionally with respect to $\mathcal{F}_0$ if

$$\int \Delta\big(\Psi, F_n(\omega)\big) P(\mathrm{d}\omega) \to 0 \quad \text{as } n \to \infty$$

where Δ denotes the Prokhorov metric for $D[0,1]$ and by $F_n(\omega)$ we denote the distribution with respect to the regular conditional probabilities P_ω (with respect to the σ-field $\mathcal{F}_0$).

We say that $S_n(t)$ converge in law to $\Psi(\omega)$ conditionally with respect to $\mathcal{I}$ if $F_n(\omega)$ denote the distributions with respect to the regular conditional probabilities P_ω with respect to the σ-field $\mathcal{I}$ and

$$\int \Delta\big(\Psi(\omega), F_n(\omega)\big) P(\mathrm{d}\omega) \to 0 \quad \text{as } n \to \infty$$

where $\Psi(\omega)$ is a probability law and the mapping $\omega \mapsto \Psi(\omega)$ is $\mathcal{I}$-measurable.

Remark that in [9], the notion of a conditional central limit theorem was defined for Markov chains; instead of $F_n(\omega)$ they used $F_n(x)$, the distributions with respect to the probabilities P^x for the chain starting at point x. Another (slightly different) definition of the conditional central limit theorem was given by Dedecker and Merlevède in [1].

It is easy to see that if the CLT holds for almost all ergodic components P_ω of P, we get the $\mathcal{I}$-conditional central limit theorem, and that the conditional CLT implies the usual CLT. The inverse implications are not true, in particular it can happen that the $\mathcal{I}$-conditional CLT takes place while for almost all ergodic components P_ω of the invariant measure any convergence fails to hold; in the Markov chain setting, we can have a conditional CLT for a (non-ergodic) Markov chain with a stationary probability measure P, which fails to hold for almost all (P) probabilities P^x where x are starting points of the chain [4].

In [9, Corollary 3], Wu and Wodroofe proved the following result:

Theorem 1. *Let P be ergodic and let $X_0 \in L^p$ for some $p > 2$ be $\mathcal{F}_0$-measurable, and*

$$\left\| E(S_n|\mathcal{F}_0)\right\|_2 = \mathrm{o}\left(\frac{\sqrt{n}}{\log^q n}\right) \tag{1}$$

for a $q \geqslant 2$. Then the process of $S_n(t)$ converges in distribution to a Brownian motion in the space $D[0,1]$, conditionally with respect to $\mathcal{F}_0$.

If, moreover, $q > 5/2$, then the convergence takes place for almost all regular conditional probabilities P_ω (with respect to the σ-field $\mathcal{F}_0$).

Wu and Wodroofe stated their result for Markov chains; in [9], in their setting $f \circ T^k = g(Z_k)$ where (Z_k) is a stationary Markov chain with a filtration $(\mathcal{F}_k)$.

Our aim is to prove an invariance principle for non-adapted processes. The probability P will in general be non-ergodic. In the non-ergodic case we will study the convergence in ergodic components P_ω of P; for a random variable η^2, $\Psi(\eta^2(\omega))$ denotes the normal law with zero mean and variance η^2. Using the technique of [7] we shall prove the following:

Theorem 2. *Let $X_0 \in L^p$ for some $p > 2$ and let X be regular, i.e. $E(X_0|\mathcal{F}_{-\infty}) = 0$ and $E(X_0|\mathcal{F}_\infty) = X_0$. If*

$$\left\| E(S_n|\mathcal{F}_0)\right\|_2 = \mathrm{o}\left(\frac{\sqrt{n}}{\log^q n}\right), \quad \left\| S_n - E(S_n|\mathcal{F}_n)\right\|_2 = \mathrm{o}\left(\frac{\sqrt{n}}{\log^q n}\right) \tag{2}$$

for a $q \geqslant 2$, then the process of

$$S_n(t) := \frac{1}{\sqrt{n}} S_{\lfloor nt \rfloor}$$

converges in distribution in the space $D[0,1]$ to a random variable $\eta^2 W$ where W is the standard Brownian motion defined on an enlarged probability space, and η^2 is $\mathcal{I}$ measurable, integrable and independent of W, and $\int \Delta(\eta^2(\omega)W, F_n(\omega))P(\mathrm{d}\omega) \to 0$.

If $q > 5/2$, then for almost every (P) ergodic component P_ω of P, $S_n(t)$ converge in $D[0,1]$ weakly to $\eta^2(\omega)W$, i.e. to $\Psi(\eta^2(\omega))$.

For proving Theorem 2 we shall give a non-adapted version of Lemma 5 from [9] which is of independent interest.

Proposition 1. *Suppose that (2) holds for a $q > 1$. Then there exists a martingale (M_n) with stationary increments for which*

$$\|S_n - M_n\|_2 = \mathrm{o}\left(\sqrt{n}\log^{1-q} n\right). \tag{3}$$

In [9] the proposition was proved for the case when X_0 is $\mathcal{F}_0$-measurable and the assumption (2) is reduced to (1). No ergodicity assumption is needed.

If $q = 1$ then there exists a process satisfying (1) but with no approximation by a stationary martingale difference sequence and there is no limit law (cf. [8]).

2. The proofs

Proof of Proposition 1. Let P_i denote the projection operator in L^2 defined by

$$P_i f = E(f|\mathcal{F}_i) - E(f|\mathcal{F}_{i-1}), \quad i \in \mathbb{Z},$$

$Uf = f \circ T$, $f \in L^2$. Remark that

$$UP_i = P_{i+1}U.$$

For f regular, i.e. $f \in L^2(\mathcal{F}_\infty) \ominus L^2(\mathcal{F}_{-\infty})$ (which is equivalent to $f = \sum_{i\in\mathcal{Z}} P_i f$) we define, as in [7],

$$Vf = \sum_{i\in\mathcal{Z}} U^{2i} P_{-i} f.$$

In particular we have $VP_if = P_{-i}U^{-2i}f$; V thus maps $H_i = L^2(\mathcal{F}_i) \ominus L^2(\mathcal{F}_{i-1})$ isometrically onto $H_{-i} = L^2(\mathcal{F}_{-i}) \ominus L^2(\mathcal{F}_{-i-1})$ and it is an isometry of $L^2(\mathcal{F}_\infty) \ominus L^2(\mathcal{F}_{-\infty})$ onto itself. On $L^2(\mathcal{F}_\infty) \ominus L^2(\mathcal{F}_{-\infty})$ we have

$$VU^k = U^{-k}V. \tag{4}$$

To see this we note that for $f \in L^2(\mathcal{F}_\infty) \ominus L^2(\mathcal{F}_{-\infty})$ we have

$$U^{-k}Vf = U^{-k}\sum_{i\in\mathbb{Z}} VP_if = U^{-k}\sum_{i\in\mathbb{Z}} P_{-i}U^{-2i}f = \sum_{i\in\mathbb{Z}} P_{-i-k}U^{-2i-k}f = \sum_{i\in\mathbb{Z}} VP_{i+k}U^kf = VU^k\sum_{i\in\mathbb{Z}} P_if$$

(a less general version of the equality has been proved in [7]).

We define $X_k = X'_k + X''_k$, $S'_n = \sum_{i=1}^n X'_i$, $S''_n = \sum_{i=1}^n X''_i$, where $X'_k = E(X_k|\mathcal{F}_k)$ and $X''_k = X_k - E(X_k|\mathcal{F}_k)$. By (2),

$$\big\|E(S'_n|\mathcal{F}_0)\big\|_2 = \mathrm{o}\big(\sqrt{n}\log^{-q} n\big), \qquad \big\|S''_n - E(S''_n|\mathcal{F}_{n-1})\big\|_2 = \mathrm{o}\big(\sqrt{n}\log^{-q} n\big).$$

The sequence of X'_k is adapted and satisfies (1) hence by [9, Lemma 5] there exists a martingale (M'_n) with stationary increments D'_k which, for $S'_n = \sum_{i=1}^n X'_i$, satisfies (3).

Let us define $Z_k = U^kVX''_0$. Because $VX''_0 = \sum_{i=1}^\infty P_{-i}VX''_0$, the process (Z_k) is adapted. By [7, Corollary 2(ii)] and (2), $\|E(\sum_{k=1}^{n-1} Z_k|\mathcal{F}_0)\|_2 = \|S''_n - E(S''_n|\mathcal{F}_{n-1})\|_2 = \mathrm{o}(\sqrt{n}\log^{-q} n)$ hence, by [9, L 5] there exists a martingale $(\bar{M}''_n)$ with stationary increments $\bar{D}''_k$ such that $\|\sum_{k=1}^{n-1} Z_k - \bar{M}''_n\|_2 = \mathrm{o}(\sqrt{n}\log^{1-q} n)$. By (4), $VU^k = U^{-k}V$ hence $V^{-1}Z_k = X''_{-k}$, and because $\bar{D}''_0 = P_0\bar{D}''_0$, we have $V\bar{D}''_0 = \bar{D}''_0$. Therefore,

$$V\left(\sum_{k=1}^{n-1} Z_k - \sum_{k=1}^{n-1}\bar{D}''_k\right) = \sum_{k=1}^{n-1} X''_{-k} - \sum_{k=1}^{n-1} D''_{-k}$$

where $D''_{-k} = V\bar{D}''_k$, so that

$$\left\|S''_n - \sum_{k=1}^{n-1} D''_k\right\|_2 = \mathrm{o}\left(\frac{\sqrt{n}}{\log^{q-1} n}\right)$$

where (D''_k) is a stationary martingale difference sequence. The martingale difference sequence of $D_k = D'_k + D''_k$ gives the approximation $\|S_n - \sum_{k=1}^{n-1} D_k\|_2 = \mathrm{o}(\frac{\sqrt{n}}{\log^{q-1} n})$. □

Proof of Theorem 2. By Proposition 1 there exists a stationary martingale difference sequence (D_k) such that $M_n = \sum_{k=1}^n D_k$ is an approximating martingale. Denote $R_n = S_n - M_n$, $\eta^2 = E(D_1^2|\mathcal{I})$.

In the same way as in the proof of Theorem 3 in [9] we can show that

$$P\Big(\max_{j\leqslant 2^m}|R_j| \geqslant 3\varepsilon 2^{m/2}\Big) \leqslant P\left(\max_{j,k:\,0\leqslant j,k\leqslant 2^m,\ |j-k|\leqslant a}\frac{|M_k - M_j|}{2^{m/2}} \geqslant \varepsilon\right) + P\left(\max_{j\leqslant 2^m}\frac{|X_j|}{2^{m/2}} \geqslant \frac{\varepsilon}{a}\right) + P\left(\max_{k\leqslant b}\frac{|R_{ak}|}{2^{m/2}} \geqslant \varepsilon\right), \tag{5}$$

$$P\left(\max_{j\leqslant 2^m}\frac{|X_j|}{2^{m/2}} \geqslant \frac{\varepsilon}{a}\right) \leqslant 2^mP\left(|X_1| \geqslant \frac{\varepsilon}{a}2^{m/2}\right) \leqslant \frac{a^p}{\varepsilon^p}2^{m(1-p/2)}E\big(|X_1|^p\big), \tag{6}$$

$$P\left(\max_{k\leqslant b}\frac{|R_{ak}|}{2^{m/2}} \geqslant \varepsilon\right) \leqslant \frac{d}{\varepsilon^2 2^m}\sum_{i=0}^d 2^{d-i}\mathrm{o}\big(a2^i\log^{2(1-q)}\big(a2^i\big)\big) = o\big(m^{4-2q}\big). \tag{7}$$

Let $F_n(\omega)$ denote the distribution of $S_n(t)$ (with respect to the ergodic component P_ω of P), $\Psi(\omega)$ the distribution of $\eta^2(\omega)W$ where W is the standard Brownian motion, and G_n denote the distribution of $M_n(t) := (1/\sqrt{n})M_{\lfloor nt\rfloor}$ in $D[0,1]$, $\Delta(\cdot,\cdot)$ be the Prokhorov distance. Then (cf. [9, p. 1688])

$$\Delta\big(\Psi(\omega), F_n(\omega)\big) \leqslant \Delta\big(\Psi(\omega), G_n(\omega)\big) + P_\omega\Big[\max_{k\leqslant n}|R_k| \geqslant \varepsilon\sqrt{n}\Big] + \varepsilon, \quad \varepsilon > 0.$$

Next we shall use the ideas from the proof of Corollary 3 in [9]. For the martingale difference sequence (D_k) (where $M_n = \sum_{k=1}^n D_k$) the invariance principle holds true for almost all ergodic components P_ω of P (cf. [6]). By the hypothesis $q \geqslant 2$ hence by (6) and (7),

$$P\Big[\max_{k\leqslant n} |R_k| \geqslant \varepsilon\sqrt{n}\Big] = \int P_\omega\Big[\max_{1\leqslant j\leqslant n} |R_j| \geqslant \varepsilon\sqrt{n}\Big] P(\mathrm{d}\omega) \to 0,$$

therefore $\int \Delta(\Psi(\omega), F_n(\omega)) P(\mathrm{d}\omega) \to 0$. This finishes the proof of the first part of Theorem 2.

Let $q > 5/2$. Using the Borel–Cantelli lemma we in (6) get that almost surely (P), $\max_{j\leqslant 2^m} \frac{|X_j|}{2^{m/2}} \geqslant \frac{\varepsilon}{a}$ for only finitely many m. Similarly, in (7), almost surely (P) $\max_{k\leqslant b} \frac{|R_{ak}|}{2^{m/2}} \geqslant \varepsilon$ for only finitely many m. For almost all (P) ω we thus have $P_\omega[\max_{j\leqslant 2^m} \frac{|X_j|}{2^{m/2}} \geqslant \frac{\varepsilon}{a}] \to 0$ and $P_\omega[\max_{k\leqslant b} \frac{|R_{ak}|}{2^{m/2}} \geqslant \varepsilon] \to 0$. By [6], the first term on the right-hand side of (5) converges to zero for almost all ergodic components P_ω. For almost all ergodic components P_ω of P we thus have $P_\omega[\max_{1\leqslant j\leqslant n} |R_j| \geqslant \varepsilon\sqrt{n}] \to 0$ hence $\Delta(\Psi(\omega), F_n(\omega)) \to 0$. This finishes the proof of the second part of Theorem 2. □

3. Concluding remarks

We have proved Proposition 1 by replacing the process (X''_k) by (Z_k) where $Z_k = U^k V X''_0 = V X''_{-k}$; then we used the fact that the approximation properties of (X''_k) are the same as of (Z_k). The operator V, however, in general does not correspond to any pointwise transformation.

Example. Let $\Omega = \{-1, 1\}^{\mathbb{Z}}$. For $i \in \mathbb{Z}$ let p_i be the projection of Ω into $\{-1, 1\}$. Let us show that the operator V defined by $Vf = \sum_{i=-\infty}^{\infty} U^{-i} P_0 U^{-i} f$ is not generated by a point transformation.

Suppose that there exists a point transformation T such that $Vf = f \circ T$. Because $Vp_i = p_{-i}$ for all i, $(T\omega)_i = \omega_{-i}$. Let

$$f = p_j \cdot g(p_{j-1}, \ldots, p_{j-k})$$

for some $j \in \mathbb{Z}$ and $k \geqslant 1$ where g is a function on $\{-1, 1\}^k$. We then have $f \in L^2(\mathcal{F}_j) \ominus L^2(\mathcal{F}_{j-1})$ hence

$$Vf = p_{-j} \cdot g(p_{-j-1}, \ldots, p_{-j-k})$$

while

$$f \circ T = p_{-j} \cdot g(p_{-j+1}, \ldots, p_{-j+k}).$$

The operator V does not preserve the distribution of f:

Let $f = p_0 + p_{-1} r(p_{-2})$ where $r(-1) = 0, r(1) = 1$. Then $Vf = p_0 + p_1 r(p_0)$; $f = 2$ with probability $1/8$ while $Vf = 2$ with probability $1/4$.

Therefore, there is not an immediate conclusion about an invariance principle or a law of iterated logarithm for (X''_k) when it holds for (Z_k). In particular, this concern the invariance principle of Peligrad and Utev [5]. The corresponding central limit theorem by Maxwell and Woodroofe [3] has been generalized to a non-adapted version in [7, Theorem 5].

Remark that several weak invariance principles for non-adapted processes have been given in [2].

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