Partial Differential Equations

A Bismut type theorem for subelliptic heat semigroups

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Abstract

Given a general second order subelliptic differential operator $\mathcal{L}$ defined on a vector bundle $\mathcal{E}$ over a compact manifold, we study the existence of $\lim_{t \to 0} \sigma(pt(x,x))$, where $pt$ is the heat kernel of $e^{t\mathcal{L}}$ and $\sigma$ is a linear map on $\text{End}(\mathcal{E}_x)$. Our result contains as a special case the local Atiyah–Singer index theorem for Dirac operators on Clifford bundles. Our approach is based on an extension to fiber bundles of the link pointed out by Rothschild and Stein between Nilpotent Lie groups and subelliptic heat kernel asymptotics on the diagonal.

1. Preliminaries: Carnot groups, paths integrals and strongly regular points for subelliptic operators

1.1. Carnot groups

Definition 1.1. A Carnot group of step $N$ is a simply connected Lie group $G$ whose Lie algebra can be written $\mathfrak{g} = \mathfrak{V}_1 \oplus \cdots \oplus \mathfrak{V}_N$, where $[\mathfrak{V}_i, \mathfrak{V}_j] \subset \mathfrak{V}_{i+j}$ and $\mathfrak{V}_s = 0$, for $s > N$.

On $\mathfrak{g}$ we can consider the family of linear operators $\delta_t : \mathfrak{g} \to \mathfrak{g}$, $t \geq 0$ which act by scalar multiplication $t^i$ on $\mathfrak{V}_i$. These operators are Lie algebra automorphisms due to the grading. The maps $\delta_t$ induce Lie group automorphisms $\Delta_t : G \to G$ which are called the canonical dilations of $G$. The number

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\[ D = \sum_{i=1}^{N} i \dim V_i \]
is called the homogeneous dimension of \( \mathbb{G} \).

**Definition 1.2.** A Lie group morphism \( \psi : \mathbb{G}_1 \to \mathbb{G}_2 \) is called a Carnot group morphism if
\[
\psi \circ \Delta^1_t = \Delta^2_t \circ \psi, \quad t \geq 0,
\]
where \( \Delta^1_t \) (resp. \( \Delta^2_t \)) are the canonical dilations of \( \mathbb{G}_1 \) (resp. \( \mathbb{G}_2 \)).

### 1.2. Carnot groups and path integrals

Let \( \mathbb{R}[\{X_0, \ldots, X_d\}] \) be the non commutative algebra over \( \mathbb{R} \) of the formal series with \( d + 1 \) indeterminates, that is the set of series
\[
Y = \sum_{k \geq 0} \sum_{i_1, \ldots, i_k} a_{i_1, \ldots, i_k} X_{i_1} \cdots X_{i_k}.
\]
The bracket between two elements \( U \) and \( V \) of \( \mathbb{R}[\{X_0, \ldots, X_d\}] \) is given by
\[
[U, V] = UV - VU.
\]
If \( I = (i_1, \ldots, i_k) \in \{0, \ldots, d\}^k \) is a word, we denote \( d(I) = n(I) + k \), where \( n(I) \) is the number of 0 in \( I \) and by \( X_I \) the commutator defined by \( X_I = [X_{i_1}, [X_{i_2}, \ldots, [X_{i_k-1}, X_{i_k}] \ldots] \).

For \( N \geq 1 \), let us denote \( \mathfrak{g}_{N,d} \), the Lie algebra \( \mathbb{R}[\{X_0, \ldots, X_d\}] \) quotiented by the relations
\[
\{ X_I = 0, d(I) \geq N + 1 \},
\]
and \( \pi : \mathbb{R}[\{X_0, \ldots, X_d\}] \to \mathfrak{g}_{N,d} \) the canonical surjection. We can write the stratification \( \mathfrak{g}_{N,d} = \mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_N \), where
\[
\mathcal{V}_k = \text{span}\{ \pi(X_I), d(I) = k \}.
\]
The set \( \mathfrak{g}_{N,d} \) endowed with the group law given by the Baker–Campbell–Hausdorff formula is a Carnot group that will be denoted \( \mathbb{G}_{N,d} \).

We denote by \( S_k \) the set of the permutations of \( \{0, \ldots, k\} \). If \( \sigma \in S_k \), we denote \( e(\sigma) \) the cardinality of the set \( \{ j \in \{0, \ldots, k-1\}, \sigma(j) > \sigma(j + 1) \} \), and \( \sigma(I) \) the word \( (i_{\sigma(1)}, \ldots, i_{\sigma(k)}) \).

If \( x : [0, 1] \to \mathbb{R}^d \) is an absolutely continuous path and \( I = (i_1, \ldots, i_k) \in \{0, \ldots, d\}^k \) a word, we use the notation,
\[
\Lambda_I(x)_t = \sum_{\sigma \in S_k} \frac{(-1)^{e(\sigma)}}{k^2(k-1)} \int_{0 \leq t_1 \leq \cdots \leq t_k \leq t} ds_1^{\sigma^{-1}(i_1)} \cdots ds_k^{\sigma^{-1}(i_k)},
\]
with the convention that \( x_0^0 = t \).

**Definition 1.3.** If \( x : [0, 1] \to \mathbb{R}^d \) is an absolutely continuous path,
\[
x_t^N = \sum_{I, d(I) \leq N} \Lambda_I(x)_t \pi(X_I)
\]
is called the lift of \( x \) in \( \mathbb{G}_{N,d} \).

**Remark 1.4.** The previous definition can easily be extended to Brownian paths by replacing in the definition of \( \Lambda_I(x)_t \) Riemann–Stieltjes integrals by Stratonovich integrals.

### 1.3. Strongly regular points for subelliptic operators

Let \( \mathbb{M} \) be a \( n \)-dimensional compact smooth manifold and let \( \mathcal{L} \) be an operator on \( \mathbb{M} \) that can be written
\[
\mathcal{L} = V_0 + \sum_{i=1}^{d} V_i^2.
\]
the $V_i$’s being smooth vector fields.

For $x_0 \in \mathbb{M}$, and $N \geq 1$ let us consider the smooth map

$$\pi^N_{x_0} : \mathbb{G}_{N,d} \to \mathbb{M}$$

such that for any absolutely continuous path $y : [0, 1] \to \mathbb{R}^d$

$$\pi^N_{x_0}(y_1^N) = \exp \left( \sum_{I : d(I) \leq N} A_I(y)_1 V_I \right)(x_0),$$

where $y^N$ is the lift of $y$ in $\mathbb{G}_{N,d}$. According to Chow’s theorem, such a map is well-defined in a unique way.

**Definition 1.5.** We say that $x_0 \in \mathbb{M}$ is strongly regular for $\mathcal{L}$ if there exist

(i) A Carnot group $\mathbb{G}(x_0)$;
(ii) A local diffeomorphism $\Phi_{x_0} : \mathcal{O} \to \mathcal{O}'$ ($\mathcal{O}$ is a neighborhood of 0 in $\mathbb{G}(x_0)$ and $\mathcal{O}'$ a neighborhood of $x_0$ in $\mathbb{M}$);
(iii) A surjective Carnot group morphism $\Psi : \mathbb{G}_N \to \mathbb{G}(x_0)$, for some $N \geq 1$; such that $\pi^N_{x_0} = \Phi_{x_0} \circ \Psi$.

**Remark 1.6.** If $x_0 \in \mathbb{M}$ is a strongly regular point then Hörmander’s condition is obviously satisfied at $x_0$, that is

$$\dim \text{span} \{ V_I(x_0), d(I) \leq N \} = \dim \mathbb{M}.$$  

**Proposition 1.7.** If $x_0 \in \mathbb{M}$ is a strongly regular point then, up to isomorphism, the Carnot group $\mathbb{G}(x_0)$ is unique, it shall be called the tangent space to $\mathcal{L}$ at $x_0$.

**Example 1.8 (Maximal subellipticity).** For instance, if

$$\dim \text{span} \{ V_I(x_0), d(I) \leq N \} = \dim \mathbb{M} = \dim \mathbb{G}_{N,d},$$

then, it is easily seen that $x_0$ is a strongly regular point with tangent space $\mathbb{G}_{N,d}$.

2. A Bismut type theorem for subelliptic heat semigroups

Let $\mathbb{M}$ be a compact smooth Riemannian manifold and let $\mathcal{E}$ be a finite-dimensional vector bundle over $\mathbb{M}$. We denote by $\Gamma(\mathbb{M}, \mathcal{E})$ the space of smooth sections. Let now $\nabla$ denote a connection on $\mathcal{E}$. We consider the following linear partial differential equation

$$\frac{\partial \Phi}{\partial t} = \mathcal{L} \Phi, \quad \Phi(0, x) = f(x), \quad (1)$$

where $\mathcal{L}$ is an operator on $\mathcal{E}$ that can be written $\mathcal{L} = \nabla_0 + \sum_{i=1}^d \nabla^2_i$, with $\nabla_i = \mathcal{F}_i + \nabla V_i$, $0 \leq i \leq d$, the $V_i$’s being smooth vector fields on $\mathbb{M}$ and the $\mathcal{F}_i$’s being smooth potentials (that is sections of the bundle End($\mathcal{E}$)). It is known that the solution of (1) can be written

$$\Phi(t, x) = \left( e^{t \mathcal{L}} f \right)(x) = \mathcal{P}_t f(x).$$

Let us assume that $x_0 \in \mathbb{M}$ is a strongly regular point of $\mathcal{V}_0 + \sum_{i=1}^d \nabla_i^2$ with tangent space $\mathbb{G}(x_0)$ whose homogeneous dimension is denoted $D$. From Hörmander’s theorem, there exists a smooth map $p(x_0, \cdot) : \mathbb{R}_{>0} \to \Gamma(\mathbb{M}, \text{End}(\mathcal{E}))$ such that for $\eta \in \Gamma(\mathbb{M}, \mathcal{E})$,

$$(\mathcal{P}_t \eta)(x_0) = \int_{\mathbb{M}} p_t(x_0, y) \eta(y) \, dy.$$ 

If $I \in \{0, 1, \ldots, d\}^k$ is a word, we denote $\nabla_I = [\nabla_i, [\nabla_{i_2}, \ldots, [\nabla_{i_{k-1}}, \nabla_{i_k}]] \ldots$, and $\mathcal{F}_I = \nabla_I - \nabla V_I \in \Gamma(\mathbb{M}, \text{End}(\mathcal{E}))$. If $I \in \{0, 1, \ldots, d\}^k$ is a word, we still denote $\nabla_I = [\nabla_i, [\nabla_{i_2}, \ldots, [\nabla_{i_{k-1}}, \nabla_{i_k}]] \ldots$, and $\mathcal{F}_I = \nabla_I - \nabla V_I \in \Gamma(\mathbb{M}, \text{End}(\mathcal{E}))$. 
For \( t > 0 \), let us consider the operator \( \Theta_t^N \) defined on \( \Gamma(\mathbb{M}, \mathcal{E}) \) by the property that for \( \eta \in \Gamma(\mathbb{M}, \mathcal{E}) \) and \( y \in \mathcal{O}_{x_0} \),
\[
(\Theta_t^N \eta)(y) = \mathbb{E}\left( \exp\left( \sum_{I, d(I) \leq N} \Lambda_I(B)_t \nabla_I \right) \eta \right)(x_0) \left| \sum_{I, d(I) \leq N} \Lambda_I(B)_t V_I(x_0) = 0 \right.,
\]
where \( (B_t)_{t \geq 0} \) is a \( d \)-dimensional Brownian motion.

**Lemma 2.1.** For \( t > 0 \), \( \Theta_t^N \) is a smooth section of the bundle \( \text{End}(\mathcal{E}) \).

**Proposition 2.2.** Let \( r(x_0) \) be the degree of non holonomy at \( x_0 \) (see [1] pp. 61). For \( N \geq r(x_0) \), when \( t \to 0 \),
\[
\rho_t(x_0, x_0) = q_t^N(x_0) \Theta_t^N(x_0) + O\left( t^{N+1-d} \right),
\]
where \( q_t^N(x_0) \) is the density at \( 0 \) of the random variable \( \sum_{I, d(I) \leq N} \Lambda_I(B)_t V_I(x_0) \).

Our main theorem is the following:

**Theorem 2.3.** Assume that \( \sigma_{x_0} \) is a linear map on \( \text{End}(\mathcal{E}_{x_0}) \) such that \( \sigma_{x_0}(\mathcal{F}_{I_1}(x_0) \cdots \mathcal{F}_{I_k}(x_0)) = 0 \) if \( \sum_j d(I_j) < D \). Then we have
\[
\lim_{t \to 0} \sigma_{x_0}(p_t(x_0, x_0)) = \sum C_{I_1, \ldots, I_k} \sigma_{x_0}(\mathcal{F}_{I_1}(x_0) \cdots \mathcal{F}_{I_k}(x_0)),
\]
for some universal constants \( C_{I_1, \ldots, I_k} \), where the above sum is taken over all the words \( I_1, \ldots, I_k \) such that \( \sum_{j=1}^k d(I_j) = D \).

In the elliptic case, previous corollary leads to a new proof of Atiyah–Singer local index theorem. Indeed, let us here, assume that \( \mathbb{M} \) admits a spin structure. The spin bundle \( \mathcal{S} \) over \( \mathbb{M} \) is the vector bundle such that for every \( x \in \mathbb{M} \), \( \mathcal{S}_x \) is the spinor module over the cotangent space \( T^*_x \mathbb{M} \). At each point \( x \), there is therefore a natural action of the Clifford algebra \( \text{Cl}(T^*_x \mathbb{M}) \simeq \text{End}(\mathcal{S}_x) \); this action will be denoted by \( c \). On \( \mathcal{S} \), there is a canonical elliptic first-order differential operator called the Dirac operator and denoted \( D \). In a local orthonormal frame \( e_i \), with dual frame \( e^*_i \), we have \( D = \sum_i c(e^*_i) \nabla e_i \), where \( \nabla \) is the Levi-Civita connection. By applying the above theorem to \( \mathcal{L} = -D^2 \) and \( \sigma = \text{Str} \), where \( \text{Str} \) is the supertrace, we obtain:

**Proposition 2.4.** (See [2]) For \( x \in \mathbb{M} \),
\[
\lim_{t \to 0} \text{Str} p_t(x, x) = \frac{1}{(4\pi d^2/2)^d/2} \text{Str} \mathbb{E}\left( \left( \sum_{1 \leq i < j \leq d} DR(e_i, e_j) \int_0^1 B^i_s dB^j_s - B^i_s dB^j_s \right)^{d/2} \left| B_1 = 0 \right. \right),
\]
where \( p_t \) is the heat kernel of \( e^{-tD^2} \), and \( DR(e_i, e_j) = \frac{1}{2} \sum_{1 \leq k < l \leq d} (R(e_i, e_j) e_k, e_l) e^*_k e^*_l \), with \( R \) Riemannian curvature.

**References**