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Partial Differential Equations

On the resolution of Pfaff systems in dimension three

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Abstract

We establish that the Cauchy problem associated with a Pfaff system in dimension three has a unique solution under minimal regularity assumptions on its coefficients. ***To cite this article: S. Mardare, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

Sur la résolution des systèmes de Pfaff en dimension trois. On établit que le problème de Cauchy associé à un système de Pfaff en dimension trois a une solution unique sous des hypothèses minimales de régularité sur ses coefficients. ***Pour citer cet article : S. Mardare, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Version française abrégée

Les notations sont définies dans la version anglaise. Soit Ω un ouvert connexe et simplement connexe de $\mathbb{R}^3$, soit x_0 un point de Ω, et soit Y^0 une matrice de $\mathbb{M}^{q\times\ell}$. Il est alors bien connu (voir, e.g., Thomas [5]) que le système de Pfaff

$$\partial_i Y = YA_i \quad \text{dans } \Omega,\ i \in \{1,2,3\},$$
$$Y(x^0) = Y^0,$$

admet une solution unique $Y \in \mathcal{C}^2(\Omega;\mathbb{M}^{q\times\ell})$ si les coefficients A_i appartiennent à l'espace $\mathcal{C}^1(\Omega;\mathbb{M}^\ell)$ et satisfont les conditions de compatibilité

$$\partial_j A_i - \partial_i A_j = A_iA_j - A_jA_i \quad \text{dans } \Omega \ \text{ pour tout } i<j. \tag{1}$$

L'objet de cette Note est d'établir que ce résultat reste vrai sous une hypothèse considérablement affaiblie, selon laquelle les coefficients A_i appartiennent à l'espace $L^p_{\text{loc}}(\Omega;\mathbb{M}^\ell)$, $p>3$, la condition de compatibilité ci-dessus étant alors satisfaite au sens des distributions (voir Théorème 4.1 dans la version anglaise). La preuve repose sur deux résultats principaux : un résultat de stabilité pour les systèmes de Pfaff à coefficients dans $L^p(\Omega)$ établi dans le

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Théorème 2.1 et un résultat d'approximation (sous les contraintes non linéaires (1)) des champs de matrices A_i établi dans le Théorème 3.1 de la version anglaise.

La démonstration complète de ces résultats, ainsi que leur généralisation à un domain Ω de dimension quelconque, se trouve dans [4].

1. Preliminaries

The notations $\mathbb{M}^{q\times\ell}$ and $\mathbb{M}^{\ell}$ respectively designate the set of all matrices with q rows and ℓ columns and the set of all square matrices of order ℓ. A generic point in $\mathbb{R}^3$ is denoted $x = (x_i)$ and partial derivatives are denoted $\partial_i = \frac{\partial}{\partial x_i}$. An open ball with radius R centered at $x \in \mathbb{R}^3$ is denoted $B_R(x)$, or B_R if its center is irrelevant in the subsequent analysis. The complement of a set $\Omega \subset \mathbb{R}^3$ is denoted by $\Omega^c := \mathbb{R}^3 \setminus \Omega$.

The space of distributions over an open set $\Omega \subset \mathbb{R}^3$ is denoted $\mathcal{D}'(\Omega)$. The usual Sobolev spaces being denoted $W^{m,p}(\Omega)$, we let

$$W^{m,p}_{\mathrm{loc}}(\Omega) := \left\{ f \in \mathcal{D}'(\Omega);\ f \in W^{m,p}(U) \text{ for all open set } U \Subset \Omega \right\},$$

where the notation $U \Subset \Omega$ means that the closure of U in $\mathbb{R}^3$ is a compact subset of Ω. If $p > 3$, the classes of functions in $W^{1,p}_{\mathrm{loc}}(\Omega)$ are identified with their continuous representatives, as in the Sobolev embedding theorem (see, e.g., Adams [1]). For matrix-valued and vector-valued function spaces, we shall use the notations $W^{m,p}(\Omega; \mathbb{M}^{q\times\ell})$, $W^{m,p}(\Omega; \mathbb{R}^{\ell})$, etc.

Detailed proofs of the results announced in this Note, together with their generalization to domains of arbitrary dimension, are given in [4].

2. Stability of solutions to Pfaff systems

We recall the following stability result, first established in [3], which shows that small perturbations in the L^p-norm of the coefficients of the Pfaff system and of its "initial data" induce small perturbations of its solution (in the Fréchet space $W^{1,p}_{\mathrm{loc}}$):

Theorem 2.1. *Let Ω be a connected open subset of $\mathbb{R}^3$, let $p > 3$, and let there be given sequences of matrix fields $A_i^n \in L^p(\Omega; \mathbb{M}^{\ell})$ and $Y^n \in W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{M}^{q\times\ell})$ that satisfy the Pfaff systems*

$$\partial_i Y^n = Y^n A_i^n \quad \text{in } \Omega$$

in the distributional sense. Fix a point $x^0 \in \Omega$ and assume that the sequence $(\sum_i \|A_i^n\|_{L^p(\Omega)} + \|Y^n(x^0)\|)$ is bounded from above by a constant M. Then, for any open set $K \Subset \Omega$, there exist a constant $C > 0$ (depending on M) such that, for all $n, m \in \mathbb{N}$,

$$\|Y^n - Y^m\|_{W^{1,p}(K)} \leqslant C \left\{ \sum_i \|A_i^n - A_i^m\|_{L^p(\Omega)} + \|Y^n(x^0) - Y^m(x^0)\| \right\}.$$

An immediate consequence of Theorem 2.1 is the following uniqueness result:

Corollary 2.2. *Let Ω be a connected open subset of $\mathbb{R}^3$, let $p > 3$, and let there be given matrix fields $A_i \in L^p_{\mathrm{loc}}(\Omega; \mathbb{M}^{\ell})$ and $Y, \widetilde{Y} \in W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{M}^{q\times\ell})$ that satisfy the relations*

$$\partial_i Y = Y A_i \quad \text{and} \quad \partial_i \widetilde{Y} = \widetilde{Y} A_i \quad \text{in } \mathcal{D}'(\Omega; \mathbb{M}^{q\times\ell}).$$

Assume that there exists a point $x^0 \in \Omega$ such that $Y(x^0) = \widetilde{Y}(x^0)$. Then $Y(x) = \widetilde{Y}(x)$ for all $x \in \Omega$.

3. Approximation of the Pfaff system

We show in this section that a Pfaff system with L^p_{loc}-coefficients can be approximated, in a sense specified in Theorem 3.1, with Pfaff systems with smooth coefficients. The main difficulty in establishing such a result is that the coefficients of the approximating Pfaff system must preserve the compatibility conditions, which are nonlinear.

Theorem 3.1. *Let Ω be an open subset of $\mathbb{R}^3$ and let $A_i \in L^p(\Omega;\mathbb{M}^\ell)$, $p>3$, be matrix fields that satisfy the relations*

$$\partial_j A_i - \partial_i A_j = A_i A_j - A_j A_i \quad \text{for all } i<j$$

in the space of distributions $\mathcal{D}'(\Omega;\mathbb{M}^\ell)$.

Then there exists a number $R_0 = R_0(p,\ell,\|A_i\|_{L^p(\Omega)})$ with the following property: For any open cube $\omega \Subset \Omega$ whose edges have lengths $< R_0$, there exist sequences of matrix fields $(A_i^\varepsilon)_{\varepsilon\in\mathbb{N}}$ in $\mathcal{C}^\infty(\omega;\mathbb{M}^\ell)\cap L^p(\omega;\mathbb{M}^\ell)$ that satisfy the relations

$$\partial_j A_i^\varepsilon - \partial_i A_j^\varepsilon = A_i^\varepsilon A_j^\varepsilon - A_j^\varepsilon A_i^\varepsilon \quad \text{in } \omega \text{ for all } i<j,$$
$$A_i^\varepsilon \to A_i \quad \text{in } L^p\big(\omega;\mathbb{M}^\ell\big) \text{ as } \varepsilon\to\infty.$$

Proof. The idea is to decompose the coefficients $A := (A_i):\omega\to(\mathbb{M}^\ell)^3$ into two parts,

$$A = \operatorname{Curl} X + \nabla\phi,$$

where $\phi:\omega\to\mathbb{M}^\ell$ is a vector field not subjected to any constraints, while $X=(X_i):\omega\to(\mathbb{M}^\ell)^3$ is uniquely determined by ϕ and the compatibility conditions satisfied by the coefficients A_i. Hereinafter, the gradient operator, the Curl operator, and the exterior product $\wedge$ are defined by

$$\nabla\phi := (\partial_1\phi, \partial_2\phi, \partial_3\phi),$$
$$\operatorname{Curl} X := (\partial_2 X_3 - \partial_3 X_2, \partial_3 X_1 - \partial_1 X_3, \partial_1 X_2 - \partial_2 X_1),$$
$$A\wedge B := (A_2B_3 - A_3B_2, A_3B_1 - A_1B_3, A_1B_2 - A_2B_1).$$

Then the approximation of A will be defined by mollifying ϕ with a convolution kernel, then by defining the approximation of X as the solution of a nonlinear problem that is related to the compatibility conditions that must be satisfied by the approximating coefficients. The proof is broken into several steps:

(i) *Decomposition of the coefficients.* Let there be given a cube ω satisfying the assumptions of the theorem and let ϕ and $X:=(X_i)$ be defined such that

$$A = \operatorname{Curl} X + \nabla\phi$$

and such that

$$\operatorname{div} X = 0 \quad \text{in } \omega, \qquad X_\tau = \partial_\nu X_\nu = 0 \text{ on } \partial\omega,$$

where ν is the outward-pointing unit normal to $\partial\omega$, and where X_ν and X_τ are respectively the normal and tangent components of X defined by

$$X_\nu := (X\cdot\nu)\nu = \big(\nu_i(\nu_1X_1+\nu_2X_2+\nu_3X_3)\big) \quad \text{and} \quad X_\tau := X - X_\nu.$$

Since ω is a cube, say $\omega := (a_1,b_1)\times(a_2,b_2)\times(a_3,b_3)$, these components are in fact the vector fields

$$X_\nu = (0,0,X_3) \quad \text{and} \quad X_\tau = (X_1,X_2,0) \quad \text{on } (a_1,b_1)\times(a_2,b_2)\times\{a_3,b_3\},$$

and the boundary conditions on $(a_1,b_1)\times(a_2,b_2)\times\{a_3,b_3\}$ for instance read

$$X_1 = X_2 = \partial_3X_3 = 0.$$

Note that the above equalities imply that $\partial_1X_1=\partial_2X_2=\partial_3X_3=0$ on $(a_1,b_1)\times(a_2,b_2)\times\{a_3,b_3\}$. Therefore $\operatorname{div} X = 0$ on $\partial\Omega$.

That such a decomposition exists is proved as follows: First, the problem

$$\Delta X = A\wedge A,$$
$$X_\tau = \partial_\nu X_\nu = 0 \quad \text{on } \partial\omega,$$

has a unique solution in $H^1(\omega,(\mathbb{M}^\ell)^3)$ (note that since $p>3$, $A\wedge A\in L^{p/2}(\omega,(\mathbb{M}^\ell)^3)\subset H^{-1}(\omega,(\mathbb{M}^\ell)^3)$) since it is a Laplace problem with mixed boundary conditions (the unknown X_1 satisfies Dirichlet boundary conditions on $[(a_1,b_1)\times(a_2,b_2)\times\{a_3,b_3\}]\cup[(a_1,b_1)\times\{a_2,b_2\}\times(a_3,b_3)]$ and Neumann boundary conditions on $\{a_1,b_1\}\times(a_2,b_2)\times(a_3,b_3)$, the unknowns X_2 and X_3 satisfy similar conditions on $\partial\omega$). Note that $X_i\in W^{2,p/2}(\omega;\mathbb{M}^\ell)\subset$

$W^{1,p}(\omega;\mathbb{M}^\ell)$ by the regularizing effect of the Laplace equation and by the Sobolev embeddings (we use here the assumption $p>3$). Since $\operatorname{div} X$ satisfies the system

$$\begin{aligned}&\Delta(\operatorname{div} X)=\operatorname{div}(A\wedge A)=\operatorname{div}(-\operatorname{Curl} A)=0 \quad \text{in } \omega,\\ &\operatorname{div} X=0 \quad \text{on } \partial\omega,\end{aligned}$$

we must have $\operatorname{div} X=0$ in ω. It turn, this implies that $\operatorname{Curl}\operatorname{Curl} X=-A\wedge A$, since the formula $\operatorname{Curl}\operatorname{Curl} Y=-\Delta Y+\nabla(\operatorname{div} Y)$ holds for all matrix field $Y=(Y_i)$. Hence

$$\operatorname{Curl}(\operatorname{Curl} X-A)=\operatorname{Curl}\operatorname{Curl} X+A\wedge A=0,$$

which in turn implies that there exists a field $\phi\in W^{1,p}(\omega;\mathbb{M}^\ell)$ such that $\operatorname{Curl} X-A=\nabla\phi$. This shows that the desired decomposition of A can indeed be achieved.

(ii) *Approximation of* ϕ *and* X. It suffices to mollify ϕ by convolution, i.e., $\phi^\varepsilon=\phi*\rho_\varepsilon$ with (ρ_ε) a sequence of mollifiers, then to define $X^\varepsilon\in W^{2,p/2}(\omega;\mathbb{M}^\ell)$ as the solution, given by the inverse function theorem (the assumptions of this theorem are indeed satisfied), to the nonlinear system

$$\begin{aligned}&\Delta X^\varepsilon=\big(\operatorname{Curl} X^\varepsilon+\nabla\phi^\varepsilon\big)\wedge\big(\operatorname{Curl} X^\varepsilon+\nabla\phi^\varepsilon\big) \quad \text{in } \mathcal{D}'\big(\omega;\big(\mathbb{M}^\ell\big)^3\big),\\ &X^\varepsilon_\tau=\partial_\nu X^\varepsilon_\nu=0 \quad \text{on } \partial\omega.\end{aligned}$$

Assume for a moment that we have proved that the solution to this system satisfies $\operatorname{div} X^\varepsilon=0$ in ω. This implies that $\operatorname{Curl}\operatorname{Curl} X^\varepsilon=-\Delta X^\varepsilon$, so that the previous system shows that

$$\operatorname{Curl}\operatorname{Curl} X^\varepsilon=-\big(\operatorname{Curl} X^\varepsilon+\nabla\phi^\varepsilon\big)\wedge\big(\operatorname{Curl} X^\varepsilon+\nabla\phi^\varepsilon\big) \quad \text{in } \omega,$$

which in turn implies that

$$\operatorname{Curl}\big(\operatorname{Curl} X^\varepsilon+\nabla\phi^\varepsilon\big)=-\big(\operatorname{Curl} X^\varepsilon+\nabla\phi^\varepsilon\big)\wedge\big(\operatorname{Curl} X^\varepsilon+\nabla\phi^\varepsilon\big) \quad \text{in } \omega.$$

We now prove that $\operatorname{div} X^\varepsilon=0$ in ω. First, the boundary conditions satisfied by X^ε imply that $\operatorname{div} X^\varepsilon=0$ on the boundary $\partial\omega$. Second, $\operatorname{div} X^\varepsilon$ satisfies the equation

$$\Delta\big(\operatorname{div} X^\varepsilon\big)=\operatorname{div}\big(\big(\operatorname{Curl} X^\varepsilon+\nabla\phi^\varepsilon\big)\wedge\big(\operatorname{Curl} X^\varepsilon+\nabla\phi^\varepsilon\big)\big) \quad \text{in } \omega.$$

The key formula here is

$$\operatorname{div}(Y\wedge Z)=(\operatorname{Curl} Y)\cdot Z-Y\cdot(\operatorname{Curl} Z)$$

for all fields $Y=(Y_i)$ and $Z=(Z_i)$ in $H^1(\omega;(\mathbb{M}^\ell)^3)$ (here $P\cdot Q:=\sum_i P_iQ_i$ for $P,Q:\Omega\to(\mathbb{M}^\ell)^3$). With the notation $A^\varepsilon:=\operatorname{Curl} X^\varepsilon+\nabla\phi^\varepsilon$, it implies that

$$\begin{aligned}&\operatorname{div}\big(\big(\operatorname{Curl} X^\varepsilon+\nabla\phi^\varepsilon\big)\wedge\big(\operatorname{Curl} X^\varepsilon+\nabla\phi^\varepsilon\big)\big)\\ &\quad=\operatorname{Curl}\operatorname{Curl} X^\varepsilon\cdot A^\varepsilon-A^\varepsilon\cdot\operatorname{Curl}\operatorname{Curl} X^\varepsilon\\ &\quad=\big(-\Delta X^\varepsilon+\nabla\big(\operatorname{div} X^\varepsilon\big)\big)\cdot A^\varepsilon-A^\varepsilon\cdot\big(-\Delta X^\varepsilon+\nabla\big(\operatorname{div} X^\varepsilon\big)\big)\\ &\quad=-\big(A^\varepsilon\wedge A^\varepsilon\big)\cdot A^\varepsilon+\nabla\big(\operatorname{div} X^\varepsilon\big)\cdot A^\varepsilon+A^\varepsilon\cdot\big(A^\varepsilon\wedge A^\varepsilon\big)-A^\varepsilon\cdot\nabla\big(\operatorname{div} X^\varepsilon\big).\end{aligned}$$

Since $(A^\varepsilon\wedge A^\varepsilon)\cdot A^\varepsilon=A^\varepsilon\cdot(A^\varepsilon\wedge A^\varepsilon)$, we deduce from the previous formulas that

$$\begin{aligned}&\Delta\big(\operatorname{div} X^\varepsilon\big)=\nabla\big(\operatorname{div} X^\varepsilon\big)\cdot A^\varepsilon-A^\varepsilon\cdot\nabla\big(\operatorname{div} X^\varepsilon\big) \quad \text{in } \omega,\\ &\operatorname{div} X^\varepsilon=0 \quad \text{on } \partial\omega.\end{aligned}$$

If the cube ω is small enough, this implies that $\operatorname{div} X^\varepsilon=0$ by using the energy estimate associated with the above system combined with the convergence $A^\varepsilon\to A$ in $L^p(\omega)$ (see the next step).

(iii) *Approximation of* A. Let $A^\varepsilon=\operatorname{Curl} X^\varepsilon+\nabla\phi^\varepsilon$ in ω. Then $A^\varepsilon\to A$ in $L^p(\omega)$, since $\phi^\varepsilon\to\phi$ in $W^{1,p}(\omega)$ and $X^\varepsilon\to X$ in $W^{2,p/2}(\omega;(\mathbb{M}^\ell)^3)$ (by the inverse function theorem) and, a fortiori, in $W^{1,p}(\omega;(\mathbb{M}^\ell)^3)$ (by the Sobolev embeddings). That $\operatorname{Curl} X^\varepsilon$ belongs to $\mathcal{C}^\infty(\omega;(\mathbb{M}^\ell)^3)$ is a consequence, by a bootstrap argument, of the interior regularity of the Laplacian. □

4. Existence and uniqueness of solutions

The main result of this Note is the following existence and uniqueness result for the Cauchy problem associated with a Pfaff system with L^p_{loc}-coefficients:

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^3$ be a connected and simply connected open set, let $x^0 \in \Omega$, and let $Y^0 \in \mathbb{M}^{q\times\ell}$. Let there be given three matrix fields $A_i \in L^p_{\text{loc}}(\Omega; \mathbb{M}^\ell)$, $p > 3$, satisfying the relations*

$$\partial_j A_i - \partial_i A_j = A_i A_j - A_j A_i \quad \textit{for all } i < j,$$

in the space of distributions $\mathcal{D}'(\Omega; \mathbb{M}^\ell)$. Then the Cauchy problem

$$\begin{aligned} &\partial_i Y = Y A_i \quad \textit{in } \mathcal{D}'\big(\Omega; \mathbb{M}^{q\times\ell}\big),\\ &Y\big(x^0\big) = Y^0, \end{aligned} \tag{2}$$

has one and only one solution $Y \in W^{1,p}_{\text{loc}}(\Omega; \mathbb{M}^{q\times\ell})$.

Proof. We first prove the following local existence result: Let $\widetilde{\Omega} \Subset \Omega$. Then for each open ball $B_r := B_r(y^0) \Subset \widetilde{\Omega}$ whose radius satisfies $r < \min(\frac{1}{3}\operatorname{dist}(y^0, \widetilde{\Omega}^c), \frac{R_0}{2})$, where $R_0 = R_0(\widetilde{\Omega})$ is the number defined in Theorem 3.1, there exists a field $Y \in W^{1,p}(B_r; \mathbb{M}^{q\times\ell})$ that satisfies the Pfaff system

$$\begin{aligned} &\partial_i Y = Y A_i \quad \text{in } \mathcal{D}'\big(B_r; \mathbb{M}^{q\times\ell}\big),\\ &Y\big(y^0\big) = Z^0, \end{aligned} \tag{3}$$

where Z^0 is an arbitrary matrix in $\mathbb{M}^{q\times\ell}$. We find this solution as a limit of solutions to a sequence of Pfaff systems with smooth coefficients.

Note that the assumption on the radius of the ball $B_r(x_0)$ implies that $B_{3r}(y^0) \Subset \widetilde{\Omega}$. It is then possible to choose R such that $r < R < R_0/2$ and $B_{3R}(y^0) \Subset \widetilde{\Omega}$. Since the open cube ω_{2R} centered at y^0 with edges of length $2R$ is contained in $B_{3R}(y^0) \Subset \widetilde{\Omega}$, Theorem 3.1 shows that there exist sequences of matrix fields $A_i^n \in \mathcal{C}^\infty(\omega_{2R}; \mathbb{M}^\ell) \cap L^p(\omega_{2R}; \mathbb{M}^\ell)$ that satisfy

$$\begin{aligned} &\partial_j A_i^n - \partial_i A_j^n = A_i^n A_j^n - A_j^n A_i^n \quad \text{in } \omega_{2R},\\ &A_i^n \to A_i \quad \text{in } L^p\big(\omega_{2R}; \mathbb{M}^\ell\big) \text{ as } n \to \infty. \end{aligned}$$

Since the coefficients A_i^n are smooth, the classical result on Pfaff systems (see, e.g., Thomas [5]) shows that there exists a matrix field $Y^n \in \mathcal{C}^\infty(\omega_{2R}; \mathbb{M}^{q\times\ell})$ that satisfies

$$\begin{aligned} &\partial_i Y^n = Y^n A_i^n \quad \text{in } \omega_{2R},\\ &Y^n\big(y^0\big) = Z^0. \end{aligned} \tag{4}$$

By the stability result of Theorem 2.1, there exists a constant $C > 0$ such that

$$\big\| Y^n - Y^m \big\|_{W^{1,p}(B_r)} \leqslant C \sum_i \big\| A_i^n - A_i^m \big\|_{L^p(\omega_{2R})},$$

which means that (Y^n) is a Cauchy sequence in the space $W^{1,p}(B_r; \mathbb{M}^{q\times\ell})$. Since this space is complete, there exists a field $Y \in W^{1,p}(B_r; \mathbb{M}^{q\times\ell})$ such that

$$Y^n \to Y \quad \text{in } W^{1,p}\big(B_r; \mathbb{M}^{q\times\ell}\big) \text{ as } n \to \infty.$$

In addition, the Sobolev continuous embedding $W^{1,p}(B_r; \mathbb{M}^{q\times\ell}) \subset \mathcal{C}^0(B_r; \mathbb{M}^{q\times\ell})$ shows that $Y^n(y^0) \to Y(y^0)$ in $\mathbb{M}^{q\times\ell}$ as $n \to \infty$. By passing to the limit $n \to \infty$ in the equations of the system (4), we deduce that the field Y satisfies the Pfaff system (3).

Finally, we define a global solution to the Pfaff system (2) as in the proof of Theorem 3.1 of [2], by glueing together some sequences of local solutions along curves starting from the given point x^0. We prove that this definition is unambiguous thanks to the uniqueness result of Corollary 2.2 and to the simply-connectedness of the set Ω. That this solution is unique follows from Corollary 2.2. □

Remark 1. The minimal value of p for which the Cauchy problem (2) with coefficients $A_i \in L^p_{\mathrm{loc}}(\Omega; \mathbb{M}^\ell)$ is well posed in the distributional sense is $p > 3$. Otherwise, the initial condition does not make sense because the value of Y at x^0 cannot be defined since the solution need not be continuous. Moreover, a simple example in the scalar case (i.e., when A_i and Y are usual functions) shows that the Pfaff system alone (without the initial condition at x^0) need not have a solution in the distributional sense if $p < 3$.

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