Available online at www.sciencedirect.com
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C. R. Acad. Sci. Paris, Ser. I 344 (2007) 627–630
COMPTES RENDUS MATHEMATIQUE
http://france.elsevier.com/direct/CRASS1/

Functional Analysis

Quantized moment problem

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Received 28 February 2007; accepted 14 March 2007

Available online 9 May 2007

Presented by Gilles Pisier

Abstract

In this Note we develop the fractional space technique in the local operator space framework. As the main result we present the noncommutative Albrecht–Vasilescu extension theorem, which in turn solves the quantized moment problem. ***To cite this article: A. Dosiev, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

Sur le problème du moment quantifié. Dans cette Note nous développons la technique des espaces fractionnaires dans le cadre d'espaces d'opérateurs locaux. Le résultat principal est une variante du théorème non commutatif d'Albrecht–Vasilescu sur les extensions, lequel implique une solution du problème du moment quantifié. ***Pour citer cet article : A. Dosiev, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

This Note is devoted to a quantized moment problem within the local operator space framework [7,2]. Recall that the operator moment problem [4] is to finding a positive operator-valued measure which express an integral representation for the given unital linear mapping from the algebra of all complex-valued polynomials in several real variables into the space of all sesquilinear forms on a pre-Hilbert space. Involving the fractional space technique [1,5,6], the operator moment problem has been solved in [1]. In the present work we propose a quantization of the fractional space construction in terms of local operator systems [2]. We replace the complex-valued polynomial functions in several real variables by the elements of a local operator algebra generated by a several symmetric unbounded operators on a Hilbert space, and prove that each local operator algebra is a fractional subspace of the multinormed C^*-algebra $C_{\mathcal{E}}(\mathcal{D})$ of all noncommutative continuous functions on a quantized domain $\mathcal{D}$ with an exhaustion $\mathcal{E}$. Such quantization inherits a quantized version of the operator moment problem. Based upon the Arveson–Hahn–Banach–Wittstock theorem and the fractional space technique, we derive the existence of a quantized measure that would lead to a solution of the quantized moment problem.

1. Local operator algebras

Let $\mathcal{E} = \{H_\alpha\}_{\alpha\in\Lambda}$ be an upward filtered family of closed subspaces in a Hilbert space H such that their union $\mathcal{D} = \bigcup \mathcal{E}$ is a dense subspace in H. We say that $\mathcal{D}$ is *a quantized domain with the exhaustion* $\mathcal{E}$. Thus we give a

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1631-073X/$ – see front matter
doi:10.1016/j.crma.2007.03.022

net $\mathfrak{p} = \{P_\alpha\colon \alpha \in \Lambda\}$ of orthoprojections (onto H_α, respectively) in $\mathcal{B}(H)$, such that $\lim \mathfrak{p} = 1_H$ (SOT). The set of all unbounded operators on H is denoted by $\mathfrak{L}(H)$. By *an algebra of all noncommutative continuous functions on a quantized domain $\mathcal{D}$ with an exhaustion $\mathcal{E}$*, we mean the set: $C_{\mathcal{E}}(\mathcal{D}) = \{T \in \mathfrak{L}(H)\colon \operatorname{dom}(T) = \mathcal{D}, TP_\alpha = P_\alpha T \in \mathcal{B}(H), \alpha \in \Lambda\}$. If $\mathbb{M}_k(C_{\mathcal{E}}(\mathcal{D}))$ is the space of all $k \times k$-matrices over $C_{\mathcal{E}}(\mathcal{D})$, then $\mathbb{M}_k(C_{\mathcal{E}}(\mathcal{D})) = C_{\mathcal{E}^k}(\mathcal{D}^k)$, where $\mathcal{E}^k = \{H_\alpha^k\}_{\alpha \in \Lambda}$ (see [2]). For a matrix $T \in \mathbb{M}_k(C_{\mathcal{E}}(\mathcal{D}))$ we set $\|T\|_\alpha^{(k)} = \|T|H_\alpha^k\|$, $\alpha \in \Lambda$. The family $\{\|\cdot\|_\alpha^{(k)}\colon k \in \mathbb{N}\}$ is a matrix seminorm [3] on $C_{\mathcal{E}}(\mathcal{D})$. Therefore $C_{\mathcal{E}}(\mathcal{D})$, in particular each its subspace V, turns out to be a local operator space [3]. We say that V is *a concrete local operator space on H with support $\mathcal{D}$* [2]. Confirm also that $C_{\mathcal{E}}(\mathcal{D})$ is a multinormed C^*-algebra with the defining family of seminorms $\|\cdot\|_\alpha^{(1)}$, $\alpha \in \Lambda$. Further, each $T \in V$ has an unbounded dual T^* such that $\mathcal{D} \subseteq \operatorname{dom}(T^*)$ and $T^* P_\alpha = P_\alpha T^* \in \mathcal{B}(H_\alpha)$ for all α. We set $T^* = T^*|\mathcal{D}$ and $V^* = \{T^*\colon T \in V\} \subseteq C_{\mathcal{E}}(\mathcal{D})$. A (concrete) local operator space $V \subseteq \mathfrak{L}(H)$ is called *a local operator system on H* (see [2]) if $V = V^*$ and $I_{\mathcal{D}} \in V$. A local operator system V is said to be a *local operator algebra on H with support $\mathcal{D}$* if V is a local operator system closed with respect to the multiplication, that is, $TS \in V$ whenever $T, S \in V$. An element $T \in V$ of a local operator system V is said to be *locally positive* [2] if $T|H_\alpha \geqslant 0$ in $\mathcal{B}(H_\alpha)$ for a certain $\alpha \in \Lambda$. In this case we write $T \geqslant_\alpha 0$. We also write $T =_\alpha 0$ if $T|H_\alpha = 0$. Note that $C_{\mathcal{E}}(H) = \{T \in \mathcal{B}(H)\colon T(H_\alpha) \subseteq H_\alpha, T^*(H_\alpha) \subseteq H_\alpha, \alpha \in \Lambda\}$ is a locally bounded (see [2]) operator space on H with support H. Confirm that $C_{\mathcal{E}}(H)$ is a multinormed $*$-algebra with the defining seminorm family $\{\|\cdot\|_\alpha^{(1)}\colon \alpha \in \Lambda\}$, whose completion $\widetilde{C_{\mathcal{E}}(H)}$ is a multinormed C^*-algebra. Actually, $C_{\mathcal{E}}(H)$ is a unital C^*-algebra too, associated to the 'dominating' norm from $\mathcal{B}(H)$. The restriction mapping $C_{\mathcal{E}}(H) \to C_{\mathcal{E}}(\mathcal{D})$, $T \mapsto T|\mathcal{D}$, implements a $*$-isometric embedding, which allows to identify the algebra $\widetilde{C_{\mathcal{E}}(H)}$ with a $*$-subalgebra in $C_{\mathcal{E}}(\mathcal{D})$.

We introduce a set of all denominators $\mathfrak{M}_{\mathcal{E}} = \{m \in C_{\mathcal{E}}(H)\colon m = m^*, (m|H_\alpha)^{-1} \in \mathcal{B}(H_\alpha), \alpha \in \Lambda\}$ in the C^*-algebra $C_{\mathcal{E}}(H)$. In particular, $m|\mathcal{D}\colon \mathcal{D} \to \mathcal{D}$ is a bijection and we have a noncommutative continuous function $T(m|\mathcal{D})^{-1} \in C_{\mathcal{E}}(\mathcal{D})$ whenever $T \in C_{\mathcal{E}}(H)$, $m \in \mathfrak{M}_{\mathcal{E}}$. We set $C_{\mathcal{E}}(H)/m = \{T/m\colon T \in C_{\mathcal{E}}(H)\} \subseteq C_{\mathcal{E}}(\mathcal{D})$, which is a unital local operator space on H with support $\mathcal{D}$. We say that $C_{\mathcal{E}}(H)/m$ is a *fractional space with the denominator m*. Put $n \preceq m$ for $n, m \in \mathfrak{M}_{\mathcal{E}}$, if $n^{-1}m$ is bounded. One can easily verify that $C_{\mathcal{E}}(H)/n \subseteq C_{\mathcal{E}}(H)/m$ whenever $n \preceq m$. Let $M \subseteq \mathfrak{M}_{\mathcal{E}}$ be a unital subset of denominators, that is, $1_H \in M$. A subset $M_0 \subseteq M$ is said to be a *cofinal* if for each $n \in M$ there corresponds $m \in M_0$ such that $n \preceq m$. Suppose $\mathcal{F}_m \subseteq C_{\mathcal{E}}(H)/m$ is a subspace for each $m \in M$. An algebraic sum $\mathcal{F}_M = \sum_{m \in M} \mathcal{F}_m$ of these subspaces is said to be a *fractional subspace* if $1_H/n \in \mathcal{F}_n \subseteq \mathcal{F}_m$ whenever $n \preceq m$, $n, m \in M$. Note that $\mathcal{F}_M = \sum_{m \in M_0} \mathcal{F}_m = \mathcal{F}_{M_0}$ for each cofinal subset $M_0 \subseteq M$. The sum $C_{\mathcal{E}}(H)/M = \sum_{m \in M} C_{\mathcal{E}}(H)/m$ is an example of a fractional subspace. One may replace $C_{\mathcal{E}}(H)$ with its unital C^*-subalgebra $\mathfrak{I}_M$ containing all $n^{-1}m$, $n \preceq m$, $n, m \in M$. We say that $\mathfrak{I}_M$ is a *C^*-algebra in $C_{\mathcal{E}}(H)$ related to M*.

Theorem 1. *Let $\mathcal{D}$ be a quantized domain in a Hilbert space H with an exhaustion $\mathcal{E}$. Then,*

$$C_{\mathcal{E}}(\mathcal{D}) = \widetilde{C_{\mathcal{E}}(H)} = C_{\mathcal{E}}(H)/\mathfrak{M}_{\mathcal{E}}.$$

Moreover, if $V \subseteq \mathfrak{L}(H)$ is a local operator system on H with support $\mathcal{D}$, then V is a fractional subspace in $C_{\mathcal{E}}(\mathcal{D})$ whenever $T^*T \in V$ for each $T \in V$. In particular, each local operator algebra is a fractional space.

Note that $\mathbb{M}_k(C_{\mathcal{E}}(H)/m) = C_{\mathcal{E}^k}(H^k)/(m \otimes 1_{H^k}) \subseteq C_{\mathcal{E}^k}(\mathcal{D}^k) = C_{\mathcal{E}^k}(H^k)/\mathfrak{M}_{\mathcal{E}^k}$ by Theorem 1. If $b \in \mathbb{M}_k(C_{\mathcal{E}}(H)/m)$ then we put $\|b\|_{m,\alpha}^{(k)} = \|(b(m \otimes 1_{H^k}))|H_\alpha^k\|_{\mathcal{B}(H_\alpha^k)}$. The family $q_{m,\alpha} = \{\|\cdot\|_{m,\alpha}^{(k)}\colon k \in \mathbb{N}\}$ is a matrix seminorm on $C_{\mathcal{E}}(H)/m$ for all $\alpha \in \Lambda$. The fractional space $C_{\mathcal{E}}(H)/m$ furnished with the matrix seminorms $\{q_{m,\alpha}\colon \alpha \in \Lambda\}$ turns out to be a locally bounded operator space [2]. Let $M \subseteq \mathfrak{M}_{\mathcal{E}}$ be a unital subset of denominators. One may put on $C_{\mathcal{E}}(H)/M$ the inductive local operator topology [3, Section 8] such that all inclusions $C_{\mathcal{E}}(H)/m \to C_{\mathcal{E}}(H)/M$, $m \in M$, are matrix continuous, where each fraction space $C_{\mathcal{E}}(H)/m$ is a locally bounded operator space. We say that this is a *fractional matrix topology* on $C_{\mathcal{E}}(H)/M$.

2. The fractional positivity

Let $M \subseteq \mathfrak{M}_{\mathcal{E}}$ be a nonempty subset of denominators with its fixed cofinal subset M_0, and let $\mathfrak{I}_M$ be a unital C^*-algebra in $C_{\mathcal{E}}(H)$ related to M. Consider a fractional subspace $\mathcal{F}_M \subseteq \mathfrak{I}_M/M$. Take $\alpha \in \Lambda$, $m \in M_0$. Let $(\mathcal{F}_m)_{M_0,\alpha,+}$ be a subset in $\mathcal{F}_m$ of those b which can be written as a finite sum $b = \sum_{i=1}^k b_i$ of elements $b_i \in \mathcal{F}_{n_i}$ such that $b_i n_i \geqslant_\alpha 0$ in $\mathfrak{I}_M$ for some $n_i \in M_0$, $n_i \preceq m$, $1 \leqslant i \leqslant k$. We write $b \geqslant_\alpha 0$ in $\mathcal{F}_m$. In particular, $T/m \geqslant_\alpha 0$

in $\mathcal{F}_m$ whenever $T \geqslant_\alpha 0$ in $\mathfrak{I}_M$. Put $(\mathcal{F}_{M_0})_{\alpha,+} = \sum_{m\in M_0}(\mathcal{F}_m)_{M_0,\alpha,+}$, which is a cone in $\mathcal{F}_M$ of all M_0-*fractionally* α-*positive* elements. If $k \in \mathbb{N}$, then we have a cone $(\mathbb{M}_k(\mathfrak{I}_M)/M_0^k)_{\alpha,+} = \sum_{m\in M_0}(\mathbb{M}_k(\mathfrak{I}_M)/(m \otimes 1_{H^k}))_{M_0^k,\alpha,+}$ of M_0^k-fractionally α-positive elements in $\mathbb{M}_k(\mathfrak{I}_M/M)$, where $M_0^k = \{m \otimes 1_{H^k}\colon m \in M_0\}$.

3. The inner product mapping

Let Δ be a pre-Hilbert space with its inner product $(x, y) \mapsto \langle x, y\rangle$, $x, y \in \Delta$, and let $SF(\Delta)$ be a space of all sesquilinear forms on Δ. Consider a unital subset $M \subseteq \mathfrak{M}_{\mathcal{E}}$ and fix its unital cofinal subset M_0. Let $\mathcal{F}_M \subseteq \mathfrak{I}_M/M$ be a fractional subspace. A linear mapping $\varphi : \mathcal{F}_M \to SF(\Delta)$ is said to be *an inner product mapping* if it is unital $(\varphi(1_H/1_H)(x, y) = \langle x, y\rangle$, $x, y \in \Delta)$ and $\varphi(1_H/m)(x, x) > 0$, $x \in \Delta\backslash\{0\}$, $m \in M_0$. Thus $\varphi(1_H/m)$ is an inner product on Δ and $\|x\|_m = (\varphi(1_H/m)(x, x))^{1/2}$, $x \in \Delta$, is a norm on Δ. Let us introduce a subspace $SF_m(\Delta) = \{\theta \in SF(\Delta)\colon \|\theta\|_m < \infty\}$, where $\|\theta\|_m = \sup\{|\theta(x, y)|\colon \|x\|_m \leqslant 1, \|y\|_m \leqslant 1\}$, $m \in M_0$. We set $SF_{M_0}(\Delta) = \sum_{m\in M_0} SF_m(\Delta)$. Since $\mathbb{M}_k(SF(\Delta)) \subseteq SF(\Delta^k)$, $k \in \mathbb{N}$, each $\|\cdot\|_m$ defines a matrix gauge $\|\cdot\|_m^{(k)} = \|\cdot\|_{m\otimes 1_{H^k}}$, $k \in \mathbb{N}$, on $SF(\Delta)$. Therefore $SF_m(\Delta)$ is an operator space. In particular, $SF_{M_0}(\Delta)$ is a local operator space with the inductive matrix topology.

Let $\varphi : \mathcal{F}_M \to SF_{M_0}(\Delta)$ be an inner-product mapping, $\varphi_m = \varphi|_{\mathcal{F}_m}$, $\varphi_{m,x} : \mathcal{F}_m \to \mathbb{C}$, $\varphi_{m,x}(b) = \varphi(b)(x, x)$, $x \in \Delta$, and let $\varphi_m^{(k)}\colon \mathbb{M}_k(\mathcal{F}_m) \to \mathbb{M}_k(SF_{M_0}(\Delta))$ be the canonical extension of φ_m, $k \in \mathbb{N}$. Put $\|\varphi_m^{(k)}\|_{m,\alpha}^{(k)} = \sup\{\|\varphi_m^{(k)}(b)\|_m^{(k)}\colon \|b\|_{m,\alpha}^{(k)} \leqslant 1\}$ and $\|\varphi_m\|_{m,\alpha,cb} = \sup\{\|\varphi_m^{(k)}\|_{m,\alpha}^{(k)}\colon k \in \mathbb{N}\}, \alpha \in \Lambda$. We say that φ is *completely* α-*contractive* if $\|\varphi_m\|_{m,\alpha,cb} \leqslant 1$ for all $m \in M_0$. In particular, $\varphi(\mathcal{F}_m) \subseteq SF_m(\Delta)$ for all $m \in M_0$.

Now let $\varphi : \mathcal{F}_M \to SF_{M_0}(\Delta)$ be an inner product mapping. A matrix $\theta \in \mathbb{M}_k(SF_{M_0}(\Delta))$ is said to be *positive* if θ is positive being an element of $SF_{M_0^k}(\Delta^k)$, that is, $\theta(x, x) \geqslant 0$, $x \in \Delta^k$. We say that φ is *completely* α-*positive* if each $\varphi^{(k)}$ is locally positive (see [2]) with respect to the cone $\mathbb{M}_k(\mathcal{F}_{M_0})_{\alpha,+}$. In particular, φ_m is α-compatible ($\varphi_m(b) = 0$ whenever $b =_\alpha 0$) and $\varphi_m(b)(x, x) \geqslant 0$, $x \in \Delta$, whenever $b \in (\mathcal{F}_m)_{M_0,\alpha,+}$, $m \in M_0$. Finally, we say that a linear mapping $\Psi : \mathfrak{I}_M \to \mathcal{B}(K)$ is m-*fractionally* α-*positive* if $\Psi(T(n^{-1}m)) \geqslant 0$ in $\mathcal{B}(K)$ for all $T \geqslant_\alpha 0$ in $\mathfrak{I}_M$, and $n \in M_0$, $n \preceq m$. By analogy, Ψ is said to be m-*fractionally completely* α-*positive* if each $\Psi^{(k)} : \mathbb{M}_k(\mathfrak{I}_M) \to \mathcal{B}(K^k)$ is $(m \otimes 1_{H^k})$-fractionally α-positive, that is, $\Psi^{(k)}(T(n^{-1}m \otimes 1_{H^k})) \geqslant 0$ in $\mathcal{B}(\Delta_m^k)$ whenever $T \in \mathbb{M}_k(\mathfrak{I}_M)$, $T \geqslant_\alpha 0$, and $n \preceq m$, $n \in M_0$.

4. The quantized measures

Let $\mathfrak{I}_M$ be a C^*-algebra in $C_{\mathcal{E}}(H)$ related to a unital set of denominators M with its fixed unital cofinal subset $M_0 \subseteq M$. Fix an inner product space Δ whose completion is K. We say that it is defined a *quantized* $\mathcal{B}(K)$-*valued measure on* $\mathfrak{I}_M$ *with support in* H_α, if we have a unital completely α-positive mapping $\Psi : \mathfrak{I}_M \to \mathcal{B}(K)$ such that the α-positive functionals $\mu_{x,y} : \mathfrak{I}_M \to \mathbb{C}$, $\mu_{x,y}(T) = \langle\Psi(T)x, y\rangle$, have linear extensions $\tilde{\mu}_{x,y} : \mathfrak{I}_M/M \to \mathbb{C}$, $x, y \in \Delta$, such that the mapping $\Delta \times \Delta \to \mathbb{C}$, $(x, y) \mapsto \tilde{\mu}_{x,y}(b)$, is a sesquilinear form on Δ for each $b \in \mathfrak{I}_M/M$, and the linear mapping $\mu : \mathfrak{I}_M/M \to SF(\Delta)$, $\mu(b)(x, y) = \tilde{\mu}_{x,y}(b)$, is completely α-positive in the following sense that for each $k \in \mathbb{N}$ and $x = [x_i] \in \Delta^k$ the mapping,

$$\mu_x^{[k]} : \mathbb{M}_k(\mathfrak{I}_M/M) \longrightarrow \mathbb{M}_{k^2}, \quad \mu_x^{[k]}(b) = \begin{bmatrix} \tilde{\mu}_{x_1,x_1}^{(k)}(b) & \cdots & \tilde{\mu}_{x_k,x_1}^{(k)}(b) \\ \vdots & & \vdots \\ \tilde{\mu}_{x_1,x_k}^{(k)}(b) & \cdots & \tilde{\mu}_{x_k,x_k}^{(k)}(b) \end{bmatrix},$$

is locally positive with respect to the cone $(\mathbb{M}_k(\mathfrak{I}_M)/M_0^k)_{\alpha,+}$. We also set $\tilde{\mu}_x = \tilde{\mu}_{x,x}$, $x \in \Delta$.

Theorem 2. *Assume that $\Delta = K = C$, and $\Psi : \mathfrak{I}_M \to \mathbb{C}$ is a unital completely α-positive functional. Then Ψ determines a quantized $\mathbb{C}$-valued measure μ on $\mathfrak{I}_M$ with support in H_α if and only if Ψ extends to a completely α-positive functional $\widetilde{\Psi} : \mathfrak{I}_M/M \to \mathbb{C}$.*

Roughly speaking, $\mathfrak{I}_M/M$ is a supply of μ-measurable noncommutative functions.

Note that if μ is a quantized $\mathcal{B}(K)$-valued measure on $\mathfrak{I}_M$ with support in H_α then the relevant linear mapping $\mu : \mathfrak{I}_M/M \to SF(\Delta)$ is a completely α-positive inner product mapping.

Now let $\psi : \mathfrak{J}_M/M \to SF_{M_0}(\Delta)$ be a completely α-positive inner-product mapping. One can prove that $\psi(T/m)(x,y) = \langle \Psi_m(T)x, y\rangle_m$, $x, y \in \Delta$, for the uniquely determined unital m-fractionally completely α-positive mapping $\Psi_m : \mathfrak{J}_M \to \mathcal{B}(\Delta_m)$, $m \in M_0$. If $K = \Delta_{1_H}$ and $\Psi = \Psi_{1_H}$, then $\Psi : \mathfrak{J}_M \to \mathcal{B}(K)$ is a unital completely α-positive mapping.

Proposition 3. *If $\psi : \mathfrak{J}_M/M \to SF_{M_0}(\Delta)$ is a completely α-positive inner-product mapping, then the mapping $\Psi = \Psi_{1_H} : \mathfrak{J}_M \to B(K)$ generates a quantized $\mathcal{B}(K)$-valued measure μ on the C^*-algebra $\mathfrak{J}_M$ with support in H_α.*

Let $\phi : \mathcal{F}_M \to SF_{M_0}(\Delta)$ be an inner-product mapping with $\phi_m(T/m)(x,y) = \langle \Psi_m(T)x, y\rangle_m$ for some linear maps $\Psi_m : \mathfrak{J}_M \to \mathcal{B}(\Delta_m)$, $m \in M_0$. We say that ϕ is *a α-admissible mapping for $\mathfrak{J}_M/M$* if each Ψ_m is m-fractionally completely α-positive on $\mathfrak{J}_M$.

Proposition 4. *Let $\mathcal{F}_M \subseteq \mathfrak{J}_M/M$be a fractional subspace such that $m/n^2 \in \mathcal{F}_M$ whenever $n \preceq m$, $n, m \in M_0$. If $\phi : \mathcal{F}_M \to \mathbb{C}$ is a α-contractive functional such that $\phi(1/n)^2 = \phi(1/m)\phi(m/n^2)$ for n, $m \in M_0$, $n \preceq m$, then ϕ is α-admissible for $\mathfrak{J}_M/M$.*

Theorem 5 *(Noncommutative Albrecht–Vasilescu Theorem). Let $M \subseteq \mathfrak{M}_{\mathcal{E}}$ be a subset of denominators in $C_{\mathcal{E}}(H)$ with its unital cofinal subset M_0, $\mathcal{F}_M \subseteq \mathfrak{J}_M/M$ a fractional subspace and let $\phi : \mathcal{F}_M \to SF_{M_0}(\Delta)$ be an inner product mapping. The map ϕ extends to a unital completely α-positive mapping $\psi : \mathfrak{J}_M/M \to SF_{M_0}(\Delta)$ such that $\|\psi_{m,x}\|_{m,\alpha} = \|\phi_{m,x}\|_{m,\alpha}$ for all $m \in M_0$, $x \in \Delta$, if and only if ϕ is α-admissible for $\mathfrak{J}_M/M$. In particular, ϕ is completely α-contractive.*

5. The quantized moment problem

Fix a n-tuple $S = (S_1, \ldots, S_n)$ of mutually commuting symmetric operators in $C_{\mathcal{E}}(\mathcal{D})$ and consider the commutative set $\mathcal{S} = \{D_S^\lambda \colon \lambda \in \mathbb{Z}_+^n\}$ of denominators in $C_{\mathcal{E}}(H)$, where $D_S^\lambda = D_{S_1}^{\lambda_1} \cdots D_{S_n}^{\lambda_n}$, $D_{S_i} = 4(1+S^2)^{-1}$, $1 \leqslant i \leqslant n$. The polynomial $*$-algebra $\mathcal{P}_S$ generated by S is a fractional subspace in $\mathcal{S}'_{\mathcal{E}}/\mathcal{S}$, where $\mathcal{S}'_{\mathcal{E}}$ is the commutant of $\mathcal{S}$ in $C_{\mathcal{E}}(H)$. Consider a unital linear mapping $\phi : \mathcal{P}_S \to SF(\Delta)$. We say that ϕ is *a H_α-moment form* (*or local moment form*) if there is a quantized $\mathcal{B}(K)$-valued measure μ on $\mathcal{S}'_{\mathcal{E}}$ with support in H_α such that $\phi(p(S))(x,x) = \tilde{\mu}_x(p(S))$ for all $p(S) \in \mathcal{P}_S$ and $x \in \Delta$, where K is the completion of the inner product space Δ. In this case μ is called a representing measure for ϕ. Using Theorem 5 and Proposition 3, one may prove the following assertion:

Theorem 6. *A unital linear mapping $\phi : \mathcal{P}_S \to SF(\Delta)$ is a H_α-moment form if and only if ϕ is a completely α-contractive inner product mapping.*

Similar assertion stated in Theorem 6 for a noncommutative operator family S can be proved using Proposition 4 and Theorem 2 under the restrictive condition.

Acknowledgements

I wish to thank F.-H. Vasilescu for the useful discussions, suggestions and for sending me [1,5] and [6].

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