Functional Analysis

Quantized moment problem

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Abstract

In this Note we develop the fractional space technique in the local operator space framework. As the main result we present the noncommutative Albrecht–Vasilescu extension theorem, which in turn solves the quantized moment problem. To cite this article: A. Dosiev, C. R. Acad. Sci. Paris, Ser. I 344 (2007).

Résumé


1. Local operator algebras

Let $\mathcal{E} = \{H_{\alpha}\}_{\alpha \in \Lambda}$ be an upward filtered family of closed subspaces in a Hilbert space $H$ such that their union $\mathcal{D} = \bigcup \mathcal{E}$ is a dense subspace in $H$. We say that $\mathcal{D}$ is a quantized domain with the exhaustion $\mathcal{E}$. Such quantization inherits a quantized version of the operator moment problem. Based upon the Arveson–Hahn–Banach–Wittstock theorem and the fractional space technique, we derive the existence of a quantized measure that would lead to a solution of the quantized moment problem.
identify the algebra
fractional matrix topology

The sum $D$ is said to be a locally bounded operator space \cite{2}. Let $C(D)$ be a multinormed $*$-algebra too, associated to the ‘dominating’ norm from $B(H)$. The restriction mapping $C(H) \to C(D)$, $T \mapsto T|D$, implements a $*$-isometric embedding, which allows to identify the algebra $C(E) \otimes F$ with its $*$-subalgebra $J_M$.

We introduce a set of all denominators $\mathcal{M}_E = \{ m \in C(H) : m = m^{*}, (m|H_0)^{-1} \in B(H_0), \alpha \in \Lambda \}$ in the C*-algebra $C(H)$. In particular, $\mathcal{M}_E : \mathcal{D} \to \mathcal{D}$ is a bijection and we have a noncommutative continuous function $T(m|D)^{-1} \in C(E)(D)$ whenever $T \in C(E)(H)$, $m \in \mathcal{M}_E$. We set $C(E)(H)/m = (T/m : T \in C(E)(H)) \subseteq C(E)(D)$, which is a unital local operator space on $H$ with support $D$. We say that $C(E)(H)/m$ is a fractional space with the denominator $m$. Put $n \leq m$ for $m, n \in \mathcal{M}_E$, if $n^{-1}m$ is bounded. One can easily verify that $C(E)(H)/n \subseteq C(E)(H)/m$ whenever $n \leq m$. Let $M \subseteq \mathcal{M}_E$ be a unital subset of denominators, that is, $1_H \in M$. A subset $M_0 \subseteq M$ is said to be cofinal if for each $n \in M$ there corresponds $m \in M_0$ such that $n \leq m$. Suppose $\mathcal{F}_m \subseteq C(E)(H)/m$ is a subspace for each $m \in M$. An algebraic sum $\mathcal{F}_M = \sum_{m \in M} \mathcal{F}_m$ of these subspaces is said to be a fractional subspace if $1_H/n \in \mathcal{F}_n \subseteq \mathcal{F}_m$ whenever $n \leq m$, $n, m \in M$. Note that $\mathcal{F}_M = \sum_{m \in M_0} \mathcal{F}_m = \mathcal{F}_{M_0}$ for each cofinal subset $M_0 \subseteq M$. The sum $C(E)(H)/M = \sum_{m \in M} C(E)(H)/m$ is an example of a fractional subspace. One may replace $C(E)(H)$ with its unital C*-subalgebra $\mathcal{J}_M$ containing all $n^{-1}m, n \leq m, n, m \in M$. We say that $\mathcal{J}_M$ is a C*-algebra in $C(E)(H)$ related to $M$.

**Theorem 1.** Let $D$ be a quantized domain in a Hilbert space $H$ with an exhaustion $E$. Then,

$$C(E)(D) = C(E)(H)/C(E)(H)/\mathcal{M}_E.$$  

Moreover, if $V \subseteq \mathcal{L}(H)$ is a local operator system on $H$ with support $D$, then $V$ is a fractional subspace in $C(E)(D)$ whenever $T^*T \in V$ for each $T \in V$. In particular, each local operator algebra is a fractional space.

Note that $\mathcal{M}_k(C(E)(H)/m) = C(E)(H^k)/(m \otimes 1_H) \subseteq C(E)(D^k) = C(E)(H^k)/\mathcal{M}_E$ by Theorem 1. If $b \in \mathcal{M}_k(C(E)(H)/m)$ then we put $\|b\|^{(k)}_{m, \alpha} = \|(b(m \otimes 1_H^k))H_k\|_{B(H^k)}$. The family $\{ q_{m, \alpha} = \|b\|^{(k)}_{m, \alpha} : k \in \mathbb{N} \}$ is a matrix seminorm on $C(E)(H)/m$ for all $\alpha \in \Lambda$. The fractional space $C(E)(H)/m$ furnished with the matrix seminorms $\{ q_{m, \alpha} : \alpha \in \Lambda \}$ turns out to be a locally bounded operator space \cite{2}. Let $M \subseteq \mathcal{M}_E$ be a unital subset of denominators. One may put on $C(E)(H)/M$ the inductive local operator topology \cite[Section 8]{3} such that all inclusions $C(E)(H)/m \to C(E)(H)/M$, $m \in M$, are matrix continuous, where each fraction space $C(E)(H)/m$ is a locally bounded operator space. We say that this is a fractional matrix topology on $C(E)(H)/M$.

2. The fractional positivity

Let $M \subseteq \mathcal{M}_E$ be a nonempty subset of denominators with its fixed cofinal subset $M_0$, and let $\mathcal{J}_M$ be a unital C*-algebra in $C(E)(H)$ related to $M$. Consider a fractional subspace $\mathcal{F}_M \subseteq \mathcal{J}_M/M$. Take $\alpha \in \Lambda$, $m \in M_0$. Let $(\mathcal{F}_m \backslash \sum_{\alpha} \mathcal{F}_m)_{\alpha, m} = \{ b \in \mathcal{F}_m : b \text{ can be written as a finite sum } \sum_{\alpha=1}^{k} b_i \in \mathcal{F}_m \}$. Let $b \in \mathcal{F}_m \backslash \sum_{\alpha} \mathcal{F}_m$. Then, there is some $\alpha \in \Lambda$ and $m \in M_0$ such that $b_i n_i \geq 0$ in $\mathcal{J}_M$ for some $n_i \in M_0$, $n_i \leq m$, $1 \leq i \leq k$. We write $b \geq 0$ in $\mathcal{F}_m$. In particular, $T/m \geq 0$ in $\mathcal{F}_m$. Moreover, $C(E)(H)/\mathcal{M}_E = C(E)(H)/M$. Therefore, $C(E)(H)/\mathcal{M}_E = C(E)(H)/M$.
in $F_m$ whenever $T \geq \alpha 0$ in $J_M$. Put $(F_M)_{\alpha,+} = \sum_{m \in M_0} (F_m)_{M_0,\alpha,+}$, which is a cone in $F_M$ of all $M_0$-fractionally $\alpha$-positive elements. If $k \in \mathbb{N}$, then we have a cone $(M_0^k(J_M)/M_0^0)_{\alpha,+} = \sum_{m \in M_0} (M_0^k(J_M)/(m \otimes 1_{H^k}))_{M_0,\alpha,+}$ of $M_0^k$-fractionally $\alpha$-positive elements in $M_0^k(J_M)/M_0^0$, where $M_0^k = \{m \otimes 1_{H^k} : m \in M_0\}$.  

3. The inner product mapping

Let $\Delta$ be a pre-Hilbert space with its inner product $(x, y) \mapsto \langle x, y \rangle$, $x, y \in \Delta$, and let $SF(\Delta)$ be a space of all sesquilinear forms on $\Delta$. Consider a unital subset $M \subseteq M_\mathcal{C}$ and fix its unital cofinal subset $M_0$. Let $F_M \subseteq J_M/M$ be a fractional subspace. A linear mapping $\varphi : F_M \rightarrow SF(\Delta)$ is said to be an inner product mapping if it is unital $(\varphi(1_H/1_H)(x, y) = \langle x, y \rangle, x, y \in \Delta)$ and $\varphi(1_H/m)(x, x) > 0, x \in \Delta \setminus \{0\}, m \in M_0$. Thus $\varphi(1_H/m)$ is an inner product on $\Delta$ and $\|x\|_m = (\varphi(1_H/m)(x, x))^{1/2}, x \in \Delta$, is a norm on $\Delta$. Let us introduce a subspace $SF_0^M(\Delta) = \{\theta \in SF(\Delta) : \|\theta\|_m < \infty\}$, where $\|\theta\|_m = \sup \{\|\theta(x, y)\| : \|x\|_m \leq 1, \|y\|_m \leq 1\}, m \in M_0$. We set $SF_0^M(\Delta) = \sum_{m \in M_0} SF_0^M(\Delta)$. Since $M_0^k(SF(\Delta)) \subseteq SF(\Delta^k), k \in \mathbb{N},$ each $\|\cdot\|_m$ defines a matrix gauge $\|\cdot\|_m^k = \|\cdot\|_{m \otimes 1_{H^k}}, k \in \mathbb{N},$ on $SF(\Delta)$. Therefore $SF_0^M(\Delta)$ is an operator space. In particular, $SF_0^M(\Delta)$ is a local operator space with the inductive matrix topology.

Let $\varphi : F_M \rightarrow SF_0^M(\Delta)$ be an inner-product mapping, $\varphi_m = \varphi|_{F_m}, \varphi_{m,x} : F_m \rightarrow \mathbb{C}, \varphi_{m,x}(b) = \varphi(b)(x, x), x \in \Delta$, and let $\varphi_{m,k} : M_0^k(F_m) \rightarrow M_0^k(SF_0^M(\Delta))$ be the canonical extension of $\varphi_m, k \in \mathbb{N}$. Put $\|\varphi_m\|_{m,n}^1 = \sup\{\|\varphi_m(b)(b)\|_m^k : \|b\|_{m,n,c} \leq 1\}$ and $\|\varphi_m\|_{m,n,a,c} = \sup\{\|\varphi_m(b)(b)\|_{m,n,a,c} : k \in \mathbb{N}, \alpha \in \Lambda \}$. We say that $\varphi$ is completely $\alpha$-contractive if $\|\varphi_m\|_{m,n,a,c} \leq 1$ for all $m \in M_0$. In particular, $\varphi(F_m) \subseteq SF_0^M(\Delta)$ for all $m \in M_0$.

Now let $\varphi : F_M \rightarrow SF_0^M(\Delta)$ be an inner product mapping. A matrix $\theta \in M_0^k(SF_0^M(\Delta))$ is said to be positive if $\theta$ is positive being an element of $SF_0^M(\Delta^k)$, that is, $\langle \theta(x, x) \rangle \geq 0, x \in \Delta^k$. We say that $\varphi$ is completely $\alpha$-positive if each $\varphi_{m,k}$ is locally positive (see [2]) with respect to the cone $M_0^k(F_0^M(\Delta))$. In particular, $\varphi_m$ is $\alpha$-compatible $(\varphi_m(b) = 0$ whenever $b = 0$) and $\varphi_m(b)(x, x) \geq 0, x \in \Delta$, whenever $b \in (F_m)_{M_0,\alpha,+}, m \in M_0$. Finally, we say that a linear mapping $\varphi : J_M \rightarrow B(K)$ is $m$-fractionally $\alpha$-positive if $\varphi(T(n^{-1}m)) \in B(K)$ for all $T \geq 0$ in $J_M$, and $n \in M_0, n \leq m$. By analogy, $\varphi$ is said to be $m$-fractionally completely $\alpha$-positive if each $\varphi_{m,k} : M_0^k(J_M) \rightarrow B(K^k)$ is $\{m \otimes 1_{H^k}\}$-fractionally $\alpha$-positive, that is, $\varphi_{m,k}(T(n^{-1}m \otimes 1_{H^k})) \in B(K)$ whenever $T \in M_0^k(J_M), T \geq 0, a, n \leq m, n \in M_0$.

4. The quantized measures

Let $J_M$ be a $C^*$-algebra in $C_\mathcal{C}(H)$ related to a unital set of denominators $M$ with its fixed unital cofinal subset $M_0 \subseteq M$. Fix an inner product space $\Delta$ whose completion is $K$. We say that it is defined a quantized $B(K)$-valued measure on $J_M$ with support in $H^\alpha$, if we have a unital completely $\alpha$-positive mapping $\Psi : J_M \rightarrow B(K)$ such that the $\alpha$-positive functionals $\mu_{x,y} : J_M \rightarrow \mathbb{C}, \mu_{x,y}(T) = \langle \Psi(T) x, y \rangle$, have linear extensions $\tilde{\mu}_{x,y} : J_M/M \rightarrow \mathbb{C}, x, y \in \Delta$, such that the mapping $\Delta \times \Delta \rightarrow \mathbb{C}, (x, y) \mapsto \tilde{\mu}_{x,y}(b)$, is a sesquilinear form on $\Delta$ for each $b \in J_M/M$, and the linear mapping $\mu : J_M/M \rightarrow SF(\Delta), \mu(b)(x, y) = \tilde{\mu}_{x,y}(b)$, is completely $\alpha$-positive in the following sense that for each $k \in \mathbb{N}$ and $x = [x_i] \in \Delta^k$ the mapping,

$$
\tilde{\mu}_{x_i}^{(k)} : M_0^k(J_M/M) \rightarrow M_0^{k^2}, \quad \tilde{\mu}_{x_i}^{(k)}(b) = \begin{bmatrix}
\tilde{\mu}_{x_1,x_1}^{(k)}(b) & \cdots & \tilde{\mu}_{x_k,x_1}^{(k)}(b) \\
\vdots & & \vdots \\
\tilde{\mu}_{x_1,x_k}^{(k)}(b) & \cdots & \tilde{\mu}_{x_k,x_k}^{(k)}(b)
\end{bmatrix},
$$

is locally positive with respect to the cone $(M_0^k(J_M/M_0^0))_{\alpha,+}$. We also set $\tilde{\mu}_{x,x} = \tilde{\mu}_{x,x}, x \in \Delta$.

**Theorem 2.** Assume that $\Delta = K = \mathbb{C}$, and $\Psi : J_M \rightarrow \mathbb{C}$ is a unital completely $\alpha$-positive functional. Then $\Psi$ determines a quantized $\mathbb{C}$-valued measure $\mu$ on $\tilde{J}_M$ with support in $H^\alpha$ if and only if $\Psi$ extends to a completely $\alpha$-positive functional $\tilde{\Psi} : J_M/M \rightarrow \mathbb{C}$.

Roughly speaking, $J_M/M$ is a supply of $\mu$-measurable noncommutative functions.

Note that if $\mu$ is a quantized $B(K)$-valued measure on $J_M$ with support in $H^\alpha$ then the relevant linear mapping $\mu : J_M/M \rightarrow SF(\Delta)$ is a completely $\alpha$-positive inner product mapping.
Now let $\psi : \mathfrak{Z}_M/M \to SF_{M_0}(\Delta)$ be a completely $\alpha$-positive inner-product mapping. One can prove that $\psi(T/m)(x, y) = \langle \Psi_m(T)x, y \rangle_m$, $x, y \in \Delta$, for the uniquely determined unital $m$-fractionally completely $\alpha$-positive mapping $\Psi_m : \mathfrak{Z}_M \to \mathcal{B}(\Delta_m)$, $m \in M_0$. If $K = \Delta_{1_H}$ and $\Psi = \Psi_{1_H}$, then $\Psi : \mathfrak{Z}_M \to \mathcal{B}(K)$ is a unital completely $\alpha$-positive mapping.

**Proposition 3.** If $\psi : \mathfrak{Z}_M/M \to SF_{M_0}(\Delta)$ is a completely $\alpha$-positive inner-product mapping, then the mapping $\Psi = \Psi_{1_H} : \mathfrak{Z}_M \to \mathcal{B}(K)$ generates a quantized $\mathcal{B}(K)$-valued measure $\mu$ on the $C^*$-algebra $\mathfrak{Z}_M$ with support in $H_\alpha$.

Let $\phi : \mathcal{F}_M \to SF_{M_0}(\Delta)$ be an inner-product mapping with $\phi_m(T/m)(x, y) = \langle \Psi_m(T)x, y \rangle_m$ for some linear maps $\Psi_m : \mathfrak{Z}_M \to \mathcal{B}(\Delta_m)$, $m \in M_0$. We say that $\phi$ is a $\alpha$-admissible mapping for $\mathfrak{Z}_M/M$ if each $\Psi_m$ is $m$-fractionally completely $\alpha$-positive on $\mathfrak{Z}_M$.

**Proposition 4.** Let $\mathcal{F}_M \subseteq \mathfrak{Z}_M/M$ be a fractional subspace such that $m/n^2 \in \mathcal{F}_M$ whenever $n \leq m, n, m \in M_0$. If $\phi : \mathcal{F}_M \to \mathcal{C}$ is a $\alpha$-contractive functional such that $\phi(1/n)^2 = \phi(1/m)\phi(m/n^2)$ for $n, m \in M_0, n \leq m$, then $\phi$ is $\alpha$-admissible for $\mathfrak{Z}_M/M$.

**Theorem 5** (Noncommutative Albrecht–Vasilescu Theorem). Let $M \subseteq \mathcal{M}_\mathcal{E}$ be a subset of denominators in $C_\mathcal{E}(H)$ with its unital cofinal subset $M_0$, $\mathcal{F}_M \subseteq \mathfrak{Z}_M/M$ a fractional subspace and let $\phi : \mathcal{F}_M \to SF_{M_0}(\Delta)$ be an inner product mapping. The map $\phi$ extends to a unital completely $\alpha$-positive mapping $\Psi : \mathfrak{Z}_M/M \to SF_{M_0}(\Delta)$ such that $\|\phi_{m,x}\|_{m,a} = \|\phi_{m,x}\|_{m,a}$ for all $m \in M_0, x \in \Delta$, if and only if $\phi$ is $\alpha$-admissible for $\mathfrak{Z}_M/M$. In particular, $\phi$ is completely $\alpha$-contractive.

5. The quantized moment problem

Fix a $n$-tuple $S = (S_1, \ldots, S_n)$ of mutually commuting symmetric operators in $C_\mathcal{E}(\mathcal{D})$ and consider the commutative set $S = \{D_{S_1}^\lambda : \lambda \in \mathbb{Z}_+^n\}$ of denominators in $C_\mathcal{E}(H)$, where $D_{S_1}^\lambda = D_{S_1}^{\lambda_1} \cdots D_{S_n}^{\lambda_n}$, $D_{S_i} = 4(1 + S_i^2)^{-1}$, $1 \leq i \leq n$. The polynomial $\alpha$-algebra $\mathcal{P}_S$ generated by $S$ is a fractional subspace in $S_\mathcal{E}'/S$, where $S_\mathcal{E}'$ is the commutant of $S$ in $C_\mathcal{E}(H)$. Consider a unital linear mapping $\phi : \mathcal{P}_S \to SF(\Delta)$. We say that $\phi$ is a $H_\alpha$-moment form (or local moment form) if there is a quantized $\mathcal{B}(K)$-valued measure $\mu$ on $S_\mathcal{E}'$ with support in $H_\alpha$ such that $\phi(p(S))(x, x) = \mu_x(p(S))$ for all $p(S) \in \mathcal{P}_S$ and $x \in \Delta$, where $K$ is the completion of the inner product space $\Delta$. In this case $\mu$ is called a representing measure for $\phi$. Using Theorem 5 and Proposition 3, one may prove the following assertion:

**Theorem 6.** A unital linear mapping $\phi : \mathcal{P}_S \to SF(\Delta)$ is a $H_\alpha$-moment form if and only if $\phi$ is a completely $\alpha$-contractive inner product mapping.

Similar assertion stated in Theorem 6 for a noncommutative operator family $S$ can be proved using Proposition 4 and Theorem 2 under the restrictive condition.

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**References**