Available online at www.sciencedirect.com

ScienceDirect

C. R. Acad. Sci. Paris, Ser. I 344 (2007) 581–585

COMPTES RENDUS MATHEMATIQUE

http://france.elsevier.com/direct/CRASS1/

Algebraic Geometry/Ordinary Differential Equations

Special subschemes of the scheme of singularities of a plane foliation

Antonio Campillo [a], Jorge Olivares [b]

[a] *Departmento de Álgebra, Geometría y Topología, Universidad de Valladolid, 47005 Valladolid, Spain*
[b] *Centro de Investigación en Matemáticas, A.C. A.P. 402, Guanajuato 36000, Mexico*

Received 20 February 2007; accepted after revision 20 March 2007

Available online 24 April 2007

Presented by Bernard Malgrange

Abstract

From the fact that a foliation by curves of degree greater than one, with isolated singularities, in the complex projective plane $\mathbb{P}^2$ is uniquely determined by its subscheme of singular points (the singular subscheme of the foliation), we pose the problem of existence of proper closed subschemes Z of the singular subscheme which still determine the foliation in a unique way. We prove the existence of such subschemes Z for foliations with reduced singular subscheme. Unlike the degree $\deg Z$ of such subschemes is not sharp for the posed problem, we show that it is so in the sense that Z defines the so-called polar net of the foliation. ***To cite this article: A. Campillo, J. Olivares, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

Sous-schémas spéciaux du schéma des singularités d'un feuilletage plan. Du fait qu'un feuilletage en courbes de degré strictement supérieur à un, avec singularités isolées, dans le plan projectif complexe $\mathbb{P}^2$ est uniquement déterminé par son sous-schéma des points singuliers (le sous-schéma singulier du feuilletage), nous posons le problème de l'existence de sous-schémas fermés propres Z du sous-schéma singulier qui déterminent encore le feuilletage d'une manière unique. Nous démontrons l'existence d'un sous-schéma Z pour les feuilletages avec un sous-schéma singulier réduit. Si le degré $\deg Z$ de tels sous-schémas n'est pas optimal pour le problème posé, nous montrons qu'il en est ainsi dans le sens où Z définit ce qu'on appelle le réseau polaire du feuilletage. ***Pour citer cet article : A. Campillo, J. Olivares, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Version française abrégée

Cette Note concerne les feuilletages holomorphes en courbes (avec singularités isolées) dans le plan projectif complexe $\mathbb{P}^2$ (voir section 2 pour les définitions). Le point de départ est le résultat des auteurs disant qu'un feuilletage de degré $r \geqslant 2$ avec singularités isolées est uniquement déterminé par son sous-schéma des points singuliers (voir Proposition 2.1 ci-dessous).

E-mail addresses: campillo@agt.uva.es (A. Campillo), olivares@cimat.mx (J. Olivares).

1631-073X/$ – see front matter © 2007 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.
doi:10.1016/j.crma.2007.03.018

Étant donné un tel feuilletage $\mathcal{F}$, nous étudions le problème de l'existence de sous-schémas fermés *propres* de $\mathrm{SingS}(\mathcal{F})$ qui déterminent encore $\mathcal{F}$ d'une manière unique, à savoir de sous-schémas pour lesquels $Z \subseteq \mathrm{SingS}(\mathcal{F}')$ implique $\mathcal{F}' = \mathcal{F}$.

Notre premier résultat (Corollary 3.3) est général mais modeste.

Notre résultat principal est le Théorème 3.5 : Nous démontrons l'existence de tels sous-schémas Z, de degré $M_r = \frac{r}{2}(r+5)$, pour n'importe quel feuilletage $\mathcal{F}$ avec sous-schéma singulier *réduit*. Nous montrons sur un exemple que M_r n'est pas le degré optimal pour le problème posé, mais que c'est le cas en ce sens qu'un tel Z définit le réseau polaire du feuilletage (voir section 4).

1. Introduction

This Note deals with holomorphic foliations by curves (with isolated singularities) in the complex projective plane $\mathbb{P}^2$ (see Section 2 for definitions). The starting point is the authors' result stating that a foliation of degree $r \geqslant 2$ with isolated singularities is uniquely determined by its subscheme of singular points (see Proposition 2.1 below).

Given such a foliation $\mathcal{F}$, we study the problem of existence of *proper* closed subschemes Z of $\mathrm{SingS}(\mathcal{F})$ which still determine $\mathcal{F}$ in a unique way, namely subschemes for which $Z \subseteq \mathrm{SingS}(\mathcal{F}')$ implies $\mathcal{F}' = \mathcal{F}$. Our first result (Corollary 3.3) is general but modest. Our main result is Theorem 3.5: We prove the existence of such subschemes Z, of degree $M_r = \frac{r}{2}(r+5)$, for any foliation $\mathcal{F}$ with *reduced* singular subscheme. We show by an example that M_r is not the sharp degree for the posed problem, but that it is so in the sense that such a Z defines the polar net of the foliation (see Section 4).

2. Foliations in the projective plane

Consider the complex projective plane $\mathbb{P}^2$ and let $\mathcal{O}_{\mathbb{P}^2}$ be its structure sheaf. Let $\Theta_{\mathbb{P}^2}$, $\Omega_{\mathbb{P}^2}$ and $\mathcal{H}$ be the tangent, cotangent and hyperplane sheaves on $\mathbb{P}^2$. For an $\mathcal{O}_{\mathbb{P}^2}$-sheaf $\mathcal{E}$, we will write $\mathcal{E}(d)$ for $\mathcal{E} \otimes \mathcal{H}^{\otimes d}$, if $d \geqslant 0$ and $\mathcal{E} \otimes (\mathcal{H}^*)^{\otimes |d|}$, if $d < 0$.

A holomorphic foliation by curves with singularities (or simply a *foliation* in the sequel) of degree r on $\mathbb{P}^2$ is the class $\mathcal{F} = \hat{\alpha} \in \mathrm{Proj}\, \mathrm{H}^0(\mathbb{P}^2, \Theta_{\mathbb{P}^2}(r-1))$ of a global section $\alpha \in \mathrm{H}^0(\mathbb{P}^2, \Theta_{\mathbb{P}^2}(r-1)) = \mathrm{H}^0(\mathbb{P}^2, \mathrm{Hom}(\mathcal{H}^{\otimes(-r+1)}, \Theta_{\mathbb{P}^2})$. In homogeneous coordinates $[X, Y, Z]$ on $\mathbb{P}^2$ such global sections can be described in the following two equivalent ways:

1. In terms of homogeneous polynomial vector fields V of degree r in $\mathbb{C}^3$ ($V = V_1 \frac{\partial}{\partial X} + V_2 \frac{\partial}{\partial Y} + V_3 \frac{\partial}{\partial Z}$, with V_j homogeneous of degree r), by means of the twisted Euler sequence ([5], p. 409):
$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(r-1) \longrightarrow \mathcal{O}_{\mathbb{P}^2}(r)^{\oplus(3)} \xrightarrow{\Pi_*} \Theta_{\mathbb{P}^2}(r-1) \longrightarrow 0. \tag{1}$$
Since $\mathrm{H}^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(r-1)) = 0$, it follows from the long exact cohomology sequence associated to (1) that any $\alpha \in \mathrm{H}^0(\mathbb{P}^2, \Theta_{\mathbb{P}^2}(r-1))$ comes from some V in this way and, moreover, that any other such vector field V' defines the same α if and only if $V - V' = g \cdot R$, where g is a homogeneous polynomial of degree $r-1$, and R is the radial vector field.
2. In terms of a projective 1-form of degree $(r+1)$, which is meant to be a 1-form
$$\Omega = A\,\mathrm{d}X + B\,\mathrm{d}Y + C\,\mathrm{d}Z \tag{2}$$
where A, B and C are homogeneous polynomials of degree $r+1$ satisfying the so-called Euler's condition:
$$XA + YB + ZC = 0. \tag{3}$$

Given a vector field V defining α (in the sense of (1)), the 1-form Ω may be recovered by the equation
$$\Omega = \det\begin{pmatrix} \mathrm{d}X & \mathrm{d}Y & \mathrm{d}Z \\ X & Y & Z \\ V_1 & V_2 & V_3 \end{pmatrix}. \tag{4}$$
Conversely, it follows from [6] that every 1-form (2) satisfying (3) has the form given by (4), for some vector field V.

Let $\mathcal{F}$ be a foliation of degree r on $\mathbb{P}^2$: Its *singular subscheme* $\mathrm{SingS}(\mathcal{F})$ is the scheme of zeroes of a section $\alpha \in \mathrm{H}^0(\mathbb{P}^2, \Theta_{\mathbb{P}^2}(r-1))$ defining $\mathcal{F}$. Accordingly, the support $\mathrm{Sing}(\mathcal{F})$ of $\Gamma_0 = \mathrm{SingS}(\mathcal{F})$ and the defining ideal sheaf $\mathcal{J}_0$ of the structure sheaf $\mathcal{O}_{\Gamma_0}$ of $\mathrm{SingS}(\mathcal{F})$ will be called respectively the *singular set* and the *singular ideal* of $\mathcal{F}$. We have a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{J}_0 \longrightarrow \mathcal{O}_{\mathbb{P}^2} \longrightarrow \mathcal{O}_{\Gamma_0} \longrightarrow 0. \tag{5}$$

We shall say that $\mathcal{F}$ has *isolated singularities* if $\mathrm{Sing}(\mathcal{F})$ is zero-dimensional.

It follows from (1) that $\alpha(p) = 0$ if and only if $V(p) = g(p) \cdot p$ or $V(p) \wedge p = 0$, and from (2) and (4), that this last condition is equivalent to $A(p) = B(p) = C(p) = 0$. Hence, $\mathcal{F}$ has isolated singularities if and only if A, B and C have no common factor and, moreover, that the singular ideal $\mathcal{J}_0$ of $\mathcal{F}$ corresponds to the homogeneous ideal (A, B, C). It is clear from this description that $\mathrm{SingS}(\mathcal{F})$ is a local complete intersection subscheme of $\mathbb{P}^2$ and it is well known that

$$\deg \mathrm{SingS}(\mathcal{F}) = r^2 + r + 1$$

for $\mathcal{F}$ with isolated singularities (see [4]).

A foliation $\mathcal{F}$ is then an algebraic assignment of a tangent direction $\alpha(q)$ to each point $q \in \mathbb{P}^2 \setminus \mathrm{Sing}(\mathcal{F})$. Since $\alpha(q)$ defines a unique projective line L_q through q, this shows that $\mathcal{F}$ defines a rational map $\Phi : \mathbb{P}^2 \to \check{\mathbb{P}}^2$ called the *polarity map* of $\mathcal{F}$. The fibre $\Phi^*\ell$ of a line ℓ in $\check{\mathbb{P}}^2$ is a curve of degree $r+1$ in $\mathbb{P}^2$ and these form a 2-dimensional projective linear system of curves, called accordingly the *polar net* relative to $\mathcal{F}$ and denoted by $\Delta(\mathcal{F})$ (see [1]).

It turns out [2, Proposition 1.1] that $\Delta(\mathcal{F})$ is given by $\{\alpha A + \beta B + \gamma C = 0\colon\ [\alpha, \beta, \gamma] \in \mathbb{P}^2\}$ and hence, that its base scheme coincides with $\mathrm{SingS}(\mathcal{F})$.

Proposition 2.1. *([2, Theorem 3.5]) If $r \geqslant 2$, then there exists a unique triple A, B, C (up to a scalar multiple) in $\mathrm{H}^0(\mathbb{P}^2, \mathcal{J}_0(r+1))$ satisfying Euler's condition (3). In consequence, if $r \geqslant 2$, $\mathcal{F}$ is the unique foliation of degree r having $\mathrm{SingS}(\mathcal{F})$ as singular subscheme, and the same is true if $r = 0$.*

Remark 1. The algebraic proof of Proposition 2.1 consists on two parts: The first is to show that $\mathrm{h}^0(\mathbb{P}^2, \mathcal{J}_0(r+1)) = 3$ and the second, to show that the triple A, B, C in $\mathrm{H}^0(\mathbb{P}^2, \mathcal{J}_0(r+1))$ is the unique one satisfying (3). There is also a geometric proof to which we will refer in Section 4.

3. The proofs

The starting point is the following:

Lemma 3.1. *Let $\mathcal{F}$ be a foliation of degree $r \geqslant 2$ on $\mathbb{P}^2$, with isolated singularities. If Z is a closed subscheme of $\mathrm{SingS}(\mathcal{F})$ such that $\mathrm{h}^0(\mathbb{P}^2, \mathcal{J}_Z(r+1)) = 3$, then Z determines $\mathcal{F}$ uniquely.*

Proof. The inclusion $\mathcal{J}_0 \subset \mathcal{J}_Z$ gives an injective map

$$\mathrm{H}^0\big(\mathbb{P}^2, \mathcal{J}_0(r+1)\big) \longrightarrow \mathrm{H}^0\big(\mathbb{P}^2, \mathcal{J}_Z(r+1)\big)$$

between two vector spaces of the same dimension and is hence an isomorphism. The coefficients A, B, C of the 1-form (2) hence belong to $\mathrm{H}^0(\mathbb{P}^2, \mathcal{J}_Z(r+1))$ and they are the unique triple satisfying (3). □

Now recall that for a zero-dimensional complete intersection subscheme Γ of $\mathbb{P}^2$, given a subscheme $Z \subset \Gamma$, the *residual subscheme Z' of Z in Γ* is a subscheme $Z' \subset \Gamma$ which, among other properties, satisfies that $\deg Z + \deg Z' = \deg \Gamma$ (see [3] or [2, §1]). With this said, we state the following:

Proposition 3.2. *Let $\mathcal{F}$ be a foliation of degree $r \geqslant 2$ on $\mathbb{P}^2$, with isolated singularities. Let Z be a closed subscheme of* $\mathrm{SingS}(\mathcal{F})$ *and let Z' be its residual subscheme in* $\mathrm{SingS}(\mathcal{F})$*. Then the following conditions are equivalent*:

(i) $\mathrm{h}^0(\mathbb{P}^2, \mathcal{J}_Z(r+1)) = 3$,
(ii) $\mathrm{h}^1(\mathbb{P}^2, \mathcal{J}_Z(r+1)) = \deg Z + 3 - N_{r+1}$,

(iii) $\mathrm{h}^1(\mathbb{P}^2, \mathcal{J}_{Z'}(r-3)) = 0$,

where $N_j = h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(j)) = \binom{j+2}{2}$.

Proof. On the one hand, the long exact cohomology sequences associated to (5) for Z and Z' give

$$\begin{aligned} \mathrm{h}^0\big(\mathbb{P}^2, \mathcal{J}_Z(r+1)\big) &= \mathrm{h}^1\big(\mathbb{P}^2, \mathcal{J}_Z(r+1)\big) + N_{r+1} - \deg Z, \\ \mathrm{h}^0\big(\mathbb{P}^2, \mathcal{J}_{Z'}(r-3)\big) &= \mathrm{h}^1\big(\mathbb{P}^2, \mathcal{J}_{Z'}(r-3)\big) + N_{r-3} - \deg Z'. \end{aligned} \tag{6}$$

The first of these equations gives the equivalence between (i) and (ii).

On the other hand, consider the residual subscheme Z'' of Z in the complete intersection subscheme Γ of two generic polars. It follows that Z'' consists of Z' together with r aligned points (see [2, §1]) and hence, by applying [3, Theorem CB7] (or [2, Proposition 3.1]), Noether's $AF + BG$ and Bezout's Theorems in sequence, that

$$\begin{aligned} \mathrm{h}^1\big(\mathbb{P}^2, \mathcal{J}_Z(r+1)\big) &= \mathrm{h}^0\big(\mathbb{P}^2, \mathcal{J}_{Z''}(r-2)\big) - \mathrm{h}^0\big(\mathbb{P}^2, \mathcal{J}_\Gamma(r-2)\big) \\ &= \mathrm{h}^0\big(\mathbb{P}^2, \mathcal{J}_{Z''}(r-2)\big) = \mathrm{h}^0\big(\mathbb{P}^2, \mathcal{J}_{Z'}(r-3)\big). \end{aligned} \tag{7}$$

Eqs. (7) and (6) together give

$$\begin{aligned} \mathrm{h}^0\big(\mathbb{P}^2, \mathcal{J}_Z(r+1)\big) &= \mathrm{h}^1\big(\mathbb{P}^2, \mathcal{J}_{Z'}(r-3)\big) + N_{r-3} - \deg Z' + N_{r+1} - \deg Z \\ &= \mathrm{h}^1\big(\mathbb{P}^2, \mathcal{J}_{Z'}(r-3)\big) + 3, \end{aligned}$$

which shows the equivalence between (i) and (iii). □

These computations together with [2, Proposition 4.3] provide our first result:

Corollary 3.3. *Let $\mathcal{F}$ be a foliation of degree $r \geqslant 2$ on $\mathbb{P}^2$, with isolated singularities. Then any closed subscheme Z' of* $\mathrm{SingS}(\mathcal{F})$ *with* $\deg Z' \leqslant r-2$ *satisfies* $\mathrm{h}^1(\mathbb{P}^2, \mathcal{J}_{Z'}(r-3)) = 0$. *In consequence, every subscheme Z of* $\mathrm{SingS}(\mathcal{F})$ *with* $\deg Z \geqslant r^2 + 3$ *determines $\mathcal{F}$ uniquely.*

We shall now concentrate on the case $\deg Z = M_r = \frac{r}{2}(r+5)$.

Lemma 3.4. *Let $\mathcal{F}$ be a foliation of degree $r \geqslant 2$ on $\mathbb{P}^2$, with isolated singularities. Let Z be a closed subscheme of* $\mathrm{SingS}(\mathcal{F})$ *with* $\deg Z = M_r$ *and let Z' be its residual subscheme in* $\mathrm{SingS}(\mathcal{F})$. *Then the following conditions are equivalent*:

(i) $\mathrm{h}^0(\mathbb{P}^2, \mathcal{J}_Z(r+1)) = 3$,
(ii) $\mathrm{h}^1(\mathbb{P}^2, \mathcal{J}_Z(r+1)) = 0$,
(iii) $\mathrm{h}^1(\mathbb{P}^2, \mathcal{J}_{Z'}(r-3)) = 0$,
(iv) $\mathrm{h}^0(\mathbb{P}^2, \mathcal{J}_{Z'}(r-3)) = 0$.

Proof. The equivalence between the first three conditions is a simple consequence of Proposition 3.2, while the equivalence between (iii) and (iv) follows directly from the long exact cohomology sequence associated to (5) for Z', together with the fact that $\deg Z' = N_{r-3}$. □

Recall that condition (ii) above states that such Z imposes independent conditions on forms of degree $r+1$. Now we come to our main result:

Theorem 3.5. *Let $\mathcal{F}$ be a foliation of degree $r \geqslant 2$ on $\mathbb{P}^2$ with reduced singular subscheme $\Gamma_0 = \mathrm{SingS}(\mathcal{F})$. Then there exists a closed subscheme Z of Γ_0 with $\deg Z = M_r$ and such that $\mathrm{h}^1(\mathbb{P}^2, \mathcal{J}_Z(r+1)) = 0$. In consequence, Z determines the foliation $\mathcal{F}$ uniquely.*

Proof. Recall first form Proposition 3.2 that N_j stands for $\mathrm{h}^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(j)) = \binom{j+2}{2}$. Now let $\mathcal{J}_0$ be the singular ideal of $\mathcal{F}$. Since $\mathrm{h}^0(\mathbb{P}^2, \mathcal{J}_0(r+1)) = 3$ (see Remark 1), it follows from the long exact cohomology sequence associated to (5) that $\mathrm{h}^1(\mathbb{P}^2, \mathcal{J}_0(r+1)) = \frac{1}{2}(r-2)(r-1) = N_{r-3}$: This means that Γ_0 imposes

$$N_{r+1} - 3 = r^2 + r + 1 - N_{r-3} = \frac{r}{2}(r+5) \tag{8}$$

conditions on forms of degree $r+1$, and that precisely N_{r-3} of them are linearly dependent.

From this remark, the existence of such subschemes Z follows merely from a Linear Algebra argument: Since Γ_0 is reduced, its support consists of $r^2 + r + 1$ distinct closed points, each of which gives rise to a linear condition in the N_{r+1} coefficients of the space of forms of degree $r+1$. The rank of this system of equations is $\frac{r}{2}(r+5)$ and hence, there exists subsets of this number of conditions (and so, subsets Z of $\frac{r}{2}(r+5)$ closed points) which are linearly independent (and of which the rest of conditions are dependent) in the space of forms of degree $r+1$. Hence $\mathrm{h}^1(\mathbb{P}^2, \mathcal{J}_Z(r+1)) = 0$. This, together with Lemmas 3.4 and 3.1, give the second statement and finishes the proof. □

4. Closing remarks

Our results have the following nice interpretation: Lemma 3.1 says that $\mathrm{H}^0(\mathbb{P}^2, \mathcal{J}_Z(r+1))$ defines a net of plane curves Δ_Z which is actually $\Delta(\mathcal{F})$. Given a (closed) point $p \in \mathbb{P}^2$, two possibilities may occur: p is a base point of the linear system Δ_Z, in which case it is a singular point of the unique $\mathcal{F}$ containing Z; otherwise, the base points of the pencil $\Delta_p \subset \Delta_Z$ of curves in the net through p lie in a line L. This line is precisely $\Phi(p) = L_p$, the image of p under the polarity map (this is the heart of the geometric proof of Proposition 2.1). With this in mind, let us say that an arbitrary subscheme $Z \subset \mathbb{P}^2$

(i) *defines a net* (call it Δ_Z), if $\mathrm{h}^0(\mathbb{P}^2, I_Z(r+1)) = 3$, and that
(ii) *defines a net of polars* if it defines a net with no base curve which satisfies that, for a generic closed point p, the base points of the pencil $\Delta_p \subset \Delta_Z$ lie in a line.

The remark is that if Z is a priori a subscheme of $\mathrm{SingS}(\mathcal{F})$, then the conditions (1) Z defines a net (2) Z defines a net of polars, and (3) Z defines the net of polars $\Delta(\mathcal{F})$ of $\mathcal{F}$, are all equivalent and moreover, $\mathcal{F}$ is the unique foliation such that $Z \subset \mathrm{SingS}(\mathcal{F})$.

What we have shown then is that M_r is the minimal degree d of a subscheme $Z \subset \mathrm{SingS}(\mathcal{F})$ such that $\Delta_Z = \Delta(\mathcal{F})$. Theorem 3.5 states that there always exist subschemes with such a minimal degree for foliations $\mathcal{F}$ with $\mathrm{SingS}(\mathcal{F})$ reduced. However, this degree M_r is not sharp for uniquely determining $\mathcal{F}$ as the following example shows:

Fix a general element $\mathcal{F}_\alpha$ of the family of foliations of degree $r = 4$ given in [7], consider a subscheme $Z' \subset \mathrm{SingS}(\mathcal{F}_\alpha)$ of degree 4 lying in no line and let Z be its residual subscheme in $\mathrm{SingS}(\mathcal{F}_\alpha)$. Then $\deg Z = 17 < 18 = M_4$ and it can be shown that Z determines $\mathcal{F}_\alpha$ uniquely.

The existence problem of subschemes Z of sharp degree which determine $\mathcal{F}$ uniquely will be tackled in a forthcoming extended paper.

References

[1] A. Campillo, J. Olivares, Assigned base conditions and geometry of foliations on the projective plane, in: Singularities, Sapporo 1998, Advanced Studies in Pure Mathematics, vol. 29, 2000, pp. 97–113.
[2] A. Campillo, J. Olivares, Polarity with respect to a foliation and Cayley–Bacharach theorems, J. Reine Angew. Math. 534 (2001) 95–118.
[3] D. Eisenbud, M. Green, J. Harris, Cayley–Bacharach theorems and conjectures, Bull. Amer. Math. Soc. 33 (1996) 295–324.
[4] X. Gómez-Mont, G. Kempf, Stability of meromorphic vector fields in projective spaces, Comm. Math. Helv. 64 (1989) 462–473.
[5] Ph. Griffiths, J. Harris, Principles of Algebraic Geometry, Pure and Applied Mathematics, Wiley-Interscience (John Wiley & Sons), New York, 1978.
[6] J.P. Jouanolou, Équations de Pfaff algébriques, Lecture Notes in Mathematics, vol. 708, Springer, Berlin, 1979.
[7] A. Lins Neto, Some examples for the Poincaré and Painlevé problems, Ann. Sci. École. Norm. Sup. 35 (2002) 231–266.