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Partial Differential Equations

On instability for the cubic nonlinear Schrödinger equation

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Abstract

We study the flow map associated to the cubic, defocusing, Schrödinger equation in space dimension at least three. We consider initial data of arbitrary size in H^s, where $0 < s < s_c$, s_c the critical index, and perturbations in H^σ, where $\sigma < s_c$ is independent of s. We show an instability mechanism in some Sobolev spaces of order smaller than s. The analysis relies on two features of super-critical geometric optics: the creation of oscillation, and the ghost effect. ***To cite this article: R. Carles, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

Sur l'instabilité pour l'équation de Schrödinger avec non-linéarité cubique. Nous étudions l'équation de Schrödinger cubique défocalisante en dimension d'espace au moins trois. Pour des données initiales de taille quelconque dans H^s, $0 < s < s_c$, où s_c est l'indice critique, nous considérons des perturbations dans H^σ, avec $\sigma < s_c$ indépendant de s. On montre une instabilité dans des espaces de Sobolev d'ordre inférieur à s. La preuve repose sur une analyse de type optique géométrique en régime sur-critique. ***Pour citer cet article : R. Carles, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

1. Introduction

We consider the Cauchy problem for the cubic, defocusing Schrödinger equation:

$$\mathrm{i}\partial_t \psi + \frac{1}{2}\Delta\psi = |\psi|^2\psi, \quad x \in \mathbb{R}^n; \qquad \psi_{|t=0} = \varphi. \tag{1}$$

Formally, the mass and energy associated to this equation are independent of time:

$$\text{Mass: } M[\psi](t) = \int_{\mathbb{R}^n} \left|\psi(t,x)\right|^2 \mathrm{d}x \equiv M[\psi](0) = M[\varphi],$$

$$\text{Energy: } E[\psi](t) = \int_{\mathbb{R}^n} \left|\nabla\psi(t,x)\right|^2 \mathrm{d}x + \int_{\mathbb{R}^n} \left|\psi(t,x)\right|^4 \mathrm{d}x \equiv E[\psi](0) = E[\varphi].$$

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Scaling arguments yield the critical value for the Cauchy problem in $H^s(\mathbb{R}^n)$: $s_c = n/2-1$. Assume $n \geqslant 3$, so that $s_c > 0$. It was established in [3] that (1) is locally well-posed in $H^s(\mathbb{R}^n)$ if $s \geqslant s_c$. On the other hand, (1) is ill-posed in H^s if $s < s_c$ ([4]). Moreover, the following norm inflation phenomenon was proved in [4] (see also [1,2]): if $0 < s < s_c$, we can find $(\varphi_j)_{j\in\mathbb{N}}$ in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ with

$$\|\varphi_j\|_{H^s} \underset{j\to+\infty}{\longrightarrow} 0, \tag{2}$$

and a sequence of positive times $\tau_j \to 0$, such that the solutions ψ_j to (1) with initial data φ_j satisfy:

$$\big\|\psi_j(\tau_j)\big\|_{H^s} \underset{j\to+\infty}{\longrightarrow} +\infty.$$

In [2], this was improved to: we can find $t_j \to 0$ such that

$$\big\|\psi_j(t_j)\big\|_{H^k} \underset{j\to+\infty}{\longrightarrow} +\infty, \quad \forall k \in \left]\frac{s}{1+s_c-s}, s\right].$$

Note that (2) means that we consider the flow map near the origin. We show that inside rings of H^s, the situation is yet more involved: for data bounded in H^s, with $0 < s < s_c$, we consider perturbations which are small in H^σ for *any* $\sigma < s_c$, and infer a similar conclusion.

Theorem 1.1. *Let $n \geqslant 3$ and $0 \leqslant s < s_c = \frac{n}{2} - 1$. Fix $C_0, \delta > 0$. We can find two sequences of initial data $(\varphi_j)_{j\in\mathbb{N}}$ and $(\tilde\varphi_j)_{j\in\mathbb{N}}$ in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$, with*:

$$C_0 - \delta \leqslant \|\varphi_j\|_{H^s}, \quad \|\tilde\varphi_j\|_{H^s} \leqslant C_0 + \delta; \qquad \|\varphi_j - \tilde\varphi_j\|_{H^\sigma} \underset{j\to+\infty}{\longrightarrow} 0, \quad \forall \sigma < s_c,$$

and a sequence of positive times $t_j \to 0$, such that the solutions ψ_j and $\tilde\psi_j$ to (1)*, with initial data φ_j and $\tilde\varphi_j$ respectively, satisfy*:

$$\big\|\psi_j(t_j) - \tilde\psi_j(t_j)\big\|_{H^k} \underset{j\to+\infty}{\longrightarrow} +\infty, \quad \forall k \in \left]\frac{s}{1+s_c-s}, s\right] \quad (\textit{if } s > 0), \quad \liminf_{j\to+\infty} \big\|\psi_j(t_j) - \tilde\psi_j(t_j)\big\|_{H^{\frac{s}{1+s_c-s}}} > 0.$$

The main novelty in this result is the fact that the initial data are close to each other in H^σ, for any $\sigma < s_c$. In particular, this range for σ is independent of s.

Remark 1. As in [1,2], we consider initial data of the form

$$\varphi_j(x) = j^{n/2-s} a_0(jx),$$

for some $a_0 \in \mathcal{S}(\mathbb{R}^n)$ independent of j. The above result holds for *all* $a_0 \in \mathcal{S}(\mathbb{R}^n)$ with, say[1], $\|a_0\|_{H^s} = C_0$, and $\tilde\varphi_j(x) = (j^{n/2-s} + j)a_0(jx)$ (see Section 2)

Considering the case $s = \frac{n}{4}$, we infer from the proof of Theorem 1.1:

Corollary 1.2. *Let $n \geqslant 5$ and $C_0, \delta > 0$. We can find two sequences of initial data $(\varphi_j)_{j\in\mathbb{N}}$ and $(\tilde\varphi_j)_{j\in\mathbb{N}}$ in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$, with:*

$$C_0 - \delta \leqslant E[\varphi_j], \quad E[\tilde\varphi_j] \leqslant C_0 + \delta; \qquad M[\varphi_j] + M[\tilde\varphi_j] + E[\varphi_j - \tilde\varphi_j] \underset{j\to+\infty}{\longrightarrow} 0,$$

and a sequence of positive times $t_j \to 0$, such that the solutions ψ_j and $\tilde\psi_j$ to (1) *with initial data φ_j and $\tilde\varphi_j$ respectively, satisfy*:

$$\liminf_{j\to+\infty} E[\psi_j - \tilde\psi_j](t_j) > 0.$$

[1] Provided that we choose j sufficiently large.

2. Reduction of the problem: super-critical geometric optics

We now proceed as in [2]. We set $\varepsilon = j^{s-s_c}$: $\varepsilon \to 0$ as $j \to +\infty$. We change the unknown function:

$$u^\varepsilon(t,x) = j^{s-n/2}\psi_j\left(\frac{t}{j^{s_c+2-s}}, \frac{x}{j}\right).$$

Note that we have the relation:

$$\|\psi_j(t)\|_{\dot H^m} = j^{m-s}\|u^\varepsilon(j^{s_c+2-s}t)\|_{\dot H^m}.$$

With initial data of the form $\varphi_j(x) = j^{n/2-s}a_0(jx) + ja_1(jx)$, (1) becomes:

$$\mathrm{i}\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = |u^\varepsilon|^2 u^\varepsilon; \qquad u^\varepsilon(0,x) = a_0(x) + \varepsilon a_1(x). \tag{3}$$

We emphasize two features for the WKB analysis associated to (3). First, even if the initial datum is independent of ε, the solution instantly becomes ε-oscillatory. This is the argument of the proof of [2, Cor. 1.7]. Second, the aspect which was not used in the proof of [2, Cor. 1.7] is what was called *ghost effect* in gas dynamics ([6]): a perturbation of order ε of the initial datum may instantly become relevant at leading order. These two features are direct consequences of the fact that (3) is super-critical as far as WKB analysis is concerned (see e.g. [2]).

Consider the two solutions u^ε and $\tilde u^\varepsilon$ of (3) with $a_1 = 0$ and $a_1 = a_0$ respectively. Then Theorem 1.1 stems from the following proposition, which in turn is essentially a reformulation of [2, Prop. 1.9 and 5.1]:

Proposition 2.1. *Let $n \geqslant 1$ and $a_0 \in \mathcal{S}(\mathbb{R}^n;\mathbb{R}) \setminus \{0\}$. There exist $T > 0$ independent of $\varepsilon \in]0,1]$, and $a, \phi, \phi_1 \in C([0,T]; H^s)$ for all $s \geqslant 0$, such that*:

$$\|u^\varepsilon - a\mathrm{e}^{\mathrm{i}\phi/\varepsilon}\|_{L^\infty([0,T];H^s_\varepsilon)} + \|\tilde u^\varepsilon - a\mathrm{e}^{\mathrm{i}\phi_1}\mathrm{e}^{\mathrm{i}\phi/\varepsilon}\|_{L^\infty([0,T];H^s_\varepsilon)} = \mathcal{O}(\varepsilon), \quad \forall s \geqslant 0,$$

where $\|f\|^2_{H^s_\varepsilon} = \int_{\mathbb{R}^n}(1+|\varepsilon\xi|^2)^s|\hat f(\xi)|^2\,\mathrm{d}\xi$, and $\hat f$ stands for the Fourier transform of f. In addition, we have, in H^s:

$$\phi(t,x) = -t|a_0(x)|^2 + \mathcal{O}(t^3); \qquad \phi_1(t,x) = -2t|a_0(x)|^2 + \mathcal{O}(t^3) \quad \textit{as } t \to 0.$$

Therefore, there exists $\tau > 0$ independent of ε, such that: $\liminf_{\varepsilon\to 0}\varepsilon^s\|u^\varepsilon(\tau) - \tilde u^\varepsilon(\tau)\|_{\dot H^s} > 0$, $\forall s \geqslant 0$.

3. Outline of the proof of Proposition 2.1

The idea, due to E. Grenier [5], consists in writing the solution to (3) as $u^\varepsilon(t,x) = a^\varepsilon(t,x)\mathrm{e}^{\mathrm{i}\phi^\varepsilon(t,x)/\varepsilon}$, where a^ε is complex-valued, and ϕ^ε is real-valued. We assume that $a_0, a_1 \in \mathcal{S}(\mathbb{R}^n)$ are independent of ε. For simplicity, we also assume that they are real-valued. Impose:

$$\begin{cases} \partial_t\phi^\varepsilon + \frac{1}{2}|\nabla\phi^\varepsilon|^2 + |a^\varepsilon|^2 = 0; \quad \phi^\varepsilon(0,x) = 0. \\ \partial_t a^\varepsilon + \nabla\phi^\varepsilon\cdot\nabla a^\varepsilon + \frac{1}{2}a^\varepsilon\Delta\phi^\varepsilon = \mathrm{i}\frac{\varepsilon}{2}\Delta a^\varepsilon; \quad a^\varepsilon(0,x) = a_0(x) + \varepsilon a_1(x). \end{cases} \tag{4}$$

Working with the unknown function $\mathbf{u}^\varepsilon = {}^t(\operatorname{Re} a^\varepsilon, \operatorname{Im} a^\varepsilon, \partial_1\phi^\varepsilon, \dots, \partial_n\phi^\varepsilon)$, (4) yields a symmetric quasi-linear hyperbolic system: for $s > n/2 + 2$, there exists $T > 0$ independent of $\varepsilon \in]0,1]$ (and of s, from tame estimates), such that (4) has a unique solution $(\phi^\varepsilon, a^\varepsilon) \in C([0,T]; H^s)^2$. Moreover, the bounds in $H^s(\mathbb{R}^n)$ are independent of ε, and we see that $(\phi^\varepsilon, a^\varepsilon)$ converges to (ϕ, a), solution of:

$$\begin{cases} \partial_t\phi + \frac{1}{2}|\nabla\phi|^2 + |a|^2 = 0; \quad \phi(0,x) = 0, \\ \partial_t a + \nabla\phi\cdot\nabla a + \frac{1}{2}a\Delta\phi = 0; \quad a(0,x) = a_0(x). \end{cases}$$

More precisely, energy estimates for symmetric systems yield:

$$\|\phi^\varepsilon - \phi\|_{L^\infty([0,T];H^s)} + \|a^\varepsilon - a\|_{L^\infty([0,T];H^s)} = \mathcal{O}(\varepsilon), \quad \forall s \geqslant 0.$$

One can prove that ϕ^ε and a^ε have an asymptotic expansion in powers of ε. The next term is given by:

$$\begin{cases} \partial_t\phi^{(1)} + \nabla\phi\cdot\nabla\phi^{(1)} + 2\,\mathrm{Re}\big(\bar{a}a^{(1)}\big) = 0; \quad \phi^{(1)}|_{t=0} = 0. \\ \partial_t a^{(1)} + \nabla\phi\cdot\nabla a^{(1)} + \nabla\phi^{(1)}\cdot\nabla a + \frac{1}{2}a^{(1)}\Delta\phi + \frac{1}{2}a\Delta\phi^{(1)} = \frac{\mathrm{i}}{2}\Delta a; \quad a^{(1)}\big|_{t=0} = a_1. \end{cases}$$

Then $a^{(1)}, \phi^{(1)} \in L^\infty([0,T]; H^s)$ for every $s \geqslant 0$, and

$$\big\|a^\varepsilon - a - \varepsilon a^{(1)}\big\|_{L^\infty([0,T_*];H^s)} + \big\|\Phi^\varepsilon - \phi - \varepsilon\phi^{(1)}\big\|_{L^\infty([0,T_*];H^s)} \leqslant C_s\varepsilon^2, \quad \forall s \geqslant 0.$$

Observe that since a is real-valued, $(\phi^{(1)}, \mathrm{Re}(\bar{a}a^{(1)}))$ solves an homogeneous linear system. Therefore, if $\mathrm{Re}(\bar{a}a^{(1)}) = 0$ at time $t=0$, then $\phi^{(1)} \equiv 0$.

Considering the cases $a_1 = 0$ and $a_1 = a_0$ for u^ε and $\tilde{u}^\varepsilon$ respectively, we obtain the first assertion of Proposition 2.1. Note that the above $\mathcal{O}(\varepsilon^2)$ becomes an $\mathcal{O}(\varepsilon)$ only, since we divide ϕ^ε and ϕ by ε. This also explains why the first estimate of Proposition 2.1 is stated in H^s_ε and not in H^s. The rest of the proposition follows easily.

Remark 2. We could use the *ghost effect* at higher order. For $N \in \mathbb{N}\setminus\{0\}$, assume $\tilde{u}^\varepsilon_{|t=0} = (1+\varepsilon^N)a_0$ for instance. Then for some $\tau > 0$ independent of ε, we have

$$\liminf_{\varepsilon\to 0}\big(\varepsilon^s\big\|u^\varepsilon(\tau) - \tilde{u}^\varepsilon(\tau)\big\|_{\dot{H}^s} \times \varepsilon^{1-N}\big) > 0, \quad \forall s \geqslant 0.$$

Returning to the functions ψ, the range for k becomes:

$$k \geqslant \frac{s + (s_c - s)(N-1)}{1 + s_c - s}.$$

For this lower bound to be strictly smaller than s, we have to assume $s > N-1$.

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