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Partial Differential Equations

Existence of local strong solutions for a quasilinear Benney system

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Abstract

We prove in this Note the existence and uniqueness of a strong local solution to the Cauchy problem for the quasilinear Benney system. ***To cite this article: J.-P. Dias et al., C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

Existence d'une solution locale forte pour un système de Benney quasilinéaire. Nous prouvons dans cette Note l'existence et unicité d'une solution locale forte du problème de Cauchy pour le système de Benney quasilinéaire. ***Pour citer cet article : J.-P. Dias et al., C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

1. Introduction and main result

We consider the system introduced by Benney in [1] to study the interaction between short and long waves, for example gravity waves in fluids:

$$\begin{cases} \mathrm{i}u_t + u_{xx} = |u|^2u + vu, & \text{(a)} \\ & x \in \mathbf{R},\ t \geqslant 0, \\ v_t + \big[f(v)\big]_x = |u|^2_x, & \text{(b)} \end{cases} \tag{1}$$

where f is a polynomial real function, u and v (real) represent the short and the long wave, respectively.

In [2] the existence of weak solutions for (1) was proved for $f(v) = av^2 - bv^3$, with a and b real constants, $b > 0$, in the following sense:

Theorem 1.1. *Given $u_0, v_0 \in H^1(\mathbf{R})$ with v_0 real-valued, there exists functions*

$$u \in L^\infty\big(\mathbf{R}_+; H^1(\mathbf{R})\big), \quad v \in L^\infty\big(\mathbf{R}_+; \big(L^4 \cap L^2\big)(\mathbf{R})\big)$$

such that

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$$\mathrm{i}\int_0^\infty\!\!\int_{\mathbf{R}} u\frac{\partial\varphi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t+\int_0^\infty\!\!\int_{\mathbf{R}}\frac{\partial u}{\partial x}\frac{\partial\varphi}{\partial x}\,\mathrm{d}x\,\mathrm{d}t+\int_{\mathbf{R}} u_0(x)\varphi(x,0)\,\mathrm{d}x+\int_0^\infty\!\!\int_{\mathbf{R}}|u|^2u\varphi\,\mathrm{d}x\,\mathrm{d}t+\int_0^\infty\!\!\int_{\mathbf{R}} vu\varphi\,\mathrm{d}x\,\mathrm{d}t=0,$$

$$\int_0^\infty\!\!\int_{\mathbf{R}} v\frac{\partial\psi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t+\int_0^\infty\!\!\int_{\mathbf{R}} f(v)\frac{\partial\psi}{\partial x}\,\mathrm{d}x\,\mathrm{d}t+\int_{\mathbf{R}} v_0(x)\psi(x,0)\,\mathrm{d}x-\int_0^\infty\!\!\int_{\mathbf{R}}\frac{\partial}{\partial x}|u|^2\psi\,\mathrm{d}x\,\mathrm{d}t=0,$$

for all functions $\varphi,\psi\in C_0^1(\mathbf{R}\times[0,+\infty[)$ (*i.e. in the class of continuously differentiable functions with compact support*)*, with* φ *being complex-valued and* ψ *real-valued.*

This result was obtained for this particular system by application of the vanishing viscosity method and we could not extend the necessary estimates to the Burger's case ($a=1, b=0$) or to more general cases. Here we will prove the existence of (local) strong solutions to (1) for general f, extending previous results in [6,7] for f linear:

Theorem 1.2. *Let* $(u_0,v_0)\in H^3(\mathbf{R})\times H^2(\mathbf{R})$ *and* $f\in C^3(\mathbf{R})$*. Then there exists a unique strong solution* (u,v) *of the Cauchy problem associated to* (1)*, with*

$$(u,v)\in C^j\big([0,T];H^{3-2j}(\mathbf{R})\big)\times C^j\big([0,T];H^{2-j}(\mathbf{R})\big),\quad j=0,1.$$

Here, the life-span $T>0$ *depends exclusively on* f *and on the initial data* (u_0,v_0)*.*

The main difficulty here is the derivative-loss in the right-hand side of Eq. (1)(a). This cannot be handled easily by the Schrödinger kernel, due to its limited smoothing properties. The method employed in [6,7] for f linear, based in the inhomogeneous smoothing effect of the Schrödinger group, cannot be easily implemented for f nonlinear. We will address this problem by introducing some auxiliary functions and rewriting system (1) without derivative loss. A similar technique was introduced in [5] to solve the fully nonlinear wave equation and employed in [4], in the context of the Zakharov–Rubenchik system.

Another interesting open problem is the study of the probable blow-up of the local smooth solutions.

2. An equivalent system

Let us take (u,v) a solution of (1). By setting $F=u_t$, we obtain from (1)(a)

$$\mathrm{i}F+u_{xx}-u=|u|^2u+u(v-1),$$

and

$$u=(\Delta-1)^{-1}\big(|u|^2u+u(v-1)-\mathrm{i}F\big),\tag{2}$$

with $\Delta=\partial^2/\partial x^2$. Also, differentiating (1)(a) with respect to t leads to

$$\mathrm{i}F_t+F_{xx}=2|u|^2F+u^2\bar F+Fv+uv_t,$$

and from (1)(b),

$$\mathrm{i}F_t+F_{xx}=2|u|^2F+u^2\bar F+Fv+u|u|_x^2-uv_xf'(v).\tag{3}$$

These computations are our motivation to consider the following Cauchy problem:

$$\begin{cases}\mathrm{i}F_t+F_{xx}=2|u|^2F+u^2\bar F+Fv+u|\tilde u|_x^2-uv_xf'(v), & \text{(a)}\\ v_t+\big[f(v)\big]_x=|\tilde u|_x^2, & \text{(b)}\\ F(x,0)=F_0(x)\in H^1(\mathbf{R}),\quad v(x,0)=v_0(x)\in H^2(\mathbf{R}),\end{cases}\tag{4}$$

where u and $\tilde u$ are given in terms of F by

$$u(x,t)=u_0+\int_0^t F(x,s)\,\mathrm{d}s\quad\text{and}\quad\tilde u(x,t)=(\Delta-1)^{-1}\big(|u|^2u+u(v-1)-\mathrm{i}F\big).\tag{5}$$

Note that in this system derivative losses do not occur. Indeed, the regularization of $(\Delta - 1)^{-1}$ puts $\tilde{u}$ in H^3 and therefore the right-hand side of (4)(a) is in H^1, like F.

We will prove the following lemma:

Lemma 2.1. *Let* $(F_0, v_0) \in H^1(\mathbf{R}) \times H^2(\mathbf{R})$ *and* $f \in C^3(\mathbf{R})$. *Then there exists* $T > 0$ *and a unique strong solution* (F, v) *of the Cauchy problem* (4)(a), (b), *with*

$$(F, v) \in C^j\big([0, T]; H^{1-2j}(\mathbf{R})\big) \times C^j\big([0, T]; H^{2-j}(\mathbf{R})\big), \quad j = 0, 1.$$

Here, the life-span $T > 0$ *depends exclusively on* f *and on the initial data* (F_0, v_0).

This lemma will be proved in the next section, using the general theory of Kato for quasilinear equations [3].

We now explain why Lemma 2.1 implies our main Theorem 1.2:

If (F, v) is a solution of (4), by differentiating (5) with respect to t we obtain $u_t = F$. Replacing in (1)(a) yields by (4)(b)

$$(\mathrm{i}u_t + u_{xx})_t = 2|u|^2 F + u^2 \bar{F} + Fv + u|\tilde{u}|_x^2 - u v_x f'(v) = 2|u|^2 u_t + u^2 \bar{u}_t + u_t v + u v_t.$$

Hence $(\mathrm{i}u_t + u_{xx} - |u|^2 u - uv)_t = 0$ and we get $\mathrm{i}u_t + u_{xx} - |u|^2 u - uv = \phi_0(x)$, where $\phi_0(x) = \mathrm{i}F_0(x) + u_0''(x) - |u_0(x)|^2 u_0(x) - u_0(x) v_0(x)$. By setting

$$F_0(x) = \mathrm{i}\big(u_0''(x) - \big|u_0(x)\big|^2 u_0(x) - u_0(x) v_0(x)\big), \tag{6}$$

we obtain $\phi_0 = 0$ and (u, v) satisfies (1)(a). Furthermore, from (1)(a),

$$u = (\Delta - 1)^{-1}\big(|u|^2 u + u(v - 1) - \mathrm{i}u_t\big). \tag{7}$$

Therefore $u = \tilde{u}$ and (u, v) satisfies (1)(b). Note that $u_t = F \in C([0, T]; H^1(\mathbf{R}))$. Also

$$u(\cdot,t) = u_0(\cdot) + \int_0^t F(\cdot,s)\,\mathrm{d}s \in C\big([0, T]; H^1(\mathbf{R})\big),$$

but from (7) we have in fact $u \in C([0, T]; H^3(\mathbf{R}))$.

3. Proof of Lemma 2.1

In order to apply a variant of Theorem 6 in [3] we need to set the Cauchy problem (4) in the framework of real spaces. We introduce the new variables $F_1 = \Re F$, $F_2 = \Im F$, $u_1 = \Re u$, $u_2 = \Im u$ and, with $U = (F_1, F_2, v)$, $F_{10} = \Re F_0$, $F_{20} = \Im F_0$, (4) can be written as follows:

$$\begin{cases} \dfrac{\partial}{\partial t} U + A(U)U = g(t, U), \\ \big(F_1(x, 0), F_2(x, 0), v(x, 0)\big) = \big(F_{10}(x), F_{20}(x), v_0(x)\big) \in \big(H^1(\mathbf{R})\big)^2 \times H^2(\mathbf{R}), \end{cases} \tag{8}$$

where

$$A(U) = \begin{bmatrix} 0 & \Delta & 0 \\ -\Delta & 0 & 0 \\ 0 & 0 & f'(v)\frac{\partial}{\partial x} \end{bmatrix}$$

and

$$g(t, U) = \begin{bmatrix} 2|u^2|F_2 - (u_1^2 - u_2^2)F_2 + 2u_1 u_2 F_1 + F_2 v + u_2|\tilde{u}|_x^2 - u_2 v_x f'(v) \\ -2|u^2|F_1 - (u_1^2 - u_2^2)F_1 - 2u_1 u_2 F_2 - F_1 v - u_1|\tilde{u}|_x^2 + u_1 v_x f'(v) \\ |\tilde{u}|_x^2 \end{bmatrix}$$

which is a nonlocal source term.

Now we set $X = (H^{-1}(\mathbf{R}))^2 \times L^2(\mathbf{R})$, $Y = (H^1(\mathbf{R}))^2 \times H^2(\mathbf{R})$ and introduce $S : Y \to X$ defined by $S = (1-\Delta)I$, which is an isomorphism. Moreover $A : U = (F_1, F_2, v) \in W \to G(X, 1, \beta)$, where W is an open ball in Y centered at the origin and with radius R and $G(X, 1, \beta)$ denotes the set of all linear operators D in X such that $-D$ generates a C_0-semigroup $\{e^{-tD}\}$ with

$$\|e^{-tD}\| \leqslant e^{\beta t}, \quad t \in [0, +\infty[, \qquad \beta = \frac{1}{2}\sup_{x\in\mathbf{R}}\left|f''\big(v(x)\big)v_x(x)\right| \leqslant cR\alpha(R),$$

where $c > 0$ is a numerical constant and $\alpha(R)$ is a continuous function (cf. [3, §8]). It is easy to see that g verifies, for fixed $T > 0$, $\|g(t, U)\|_Y \leqslant \lambda$, $t \in [0, T]$, $U \in W$. Now, with $B_0(v) \in \mathcal{L}(L^2(\mathbf{R}))$, v in a ball W_1 in $H^2(\mathbf{R})$, $B_0(v)$ defined by (8.7) in [3]

$$B_0(v) = -\left[f''(v)v_{xx} + f'''(v)v_x^2\right]\frac{\partial}{\partial x}(1-\Delta)^{-1} - 2f'(v)v_x\frac{\partial^2}{\partial x^2}(1-\Delta)^{-1},$$

we introduce an operator $B(U) \in \mathcal{L}(X)$, $U = (F_1, F_2, v) \in W$, defined by

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B_0(v) \end{bmatrix}.$$

In [3, §8], Kato proved that for $v \in W_1$ we have

$$(1-\Delta)\left(f'(v)\frac{\partial}{\partial x}\right)(1-\Delta)^{-1} = f'(v)\frac{\partial}{\partial x} + B_0(v).$$

Hence, we easily derive, for $U \in W$ $SA(U)S^{-1} = A(U) + B(U)$. Now, for each pair (U, U^*), $U = (F_1, F_2, v)$ and $U^* = (F_1^*, F_2^*, v^*)$ in W we will prove that

$$\left\|g(t, U) - g(t, U^*)\right\|_{L^1(0,T';X)} \leqslant c(T') \sup_{0\leqslant t\leqslant T'}\left\|U(t) - U^*(t)\right\|_X \tag{9}$$

for $T' \in [0, T]$ where $c(T')$ is a continuous increasing function such that $c(0) = 0$.

Let us point out that if $h \in L^2(\mathbf{R})$ and $w \in H^1(\mathbf{R})$ we easily derive

$$\|hw\|_{H^{-1}} \leqslant \|h\|_{H^{-1}}\|w\|_{H^1}.$$

Hence, for example, we get, with an obvious notation, $\|F_1 u_1(u_1^* - u_1)\|_{H^{-1}} \leqslant \|F_1\|_{H^1}\|u_1\|_{H^1}\|u_1^* - u_1\|_{H^{-1}}$ and, for $t \leqslant T'$

$$\left\|f'(v)v_x\left(\int_0^t F_2\,d\tau - \int_0^t F_2^*\,d\tau\right)\right\|_{H^{-1}} \leqslant \left\|f'(v)v_x\right\|_{H^1}\int_0^t \|F - F^*\|_{H^{-1}}\,d\tau \leqslant c(T') \sup_{0\leqslant t\leqslant T'}\left\|U(t) - U^*(t)\right\|_X,$$

where $c(T')$ is a continuous increasing function such that $c(0) = 0$. Now, Lemma 2.1 is an easy consequence of Theorem 6 in [3], where the local condition (7.7) is replaced by (9) which is sufficient for the proof of this theorem.

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References

[1] D.J. Benney, A general theory for interactions between short and long waves, Stud. Appl. Math. 56 (1977) 81–94.

[2] J.P. Dias, M. Figueira, Existence of weak solutions for a quasilinear version of Benney equations, J. Hyperbolic Differential Equations, in press.

[3] T. Kato, Quasi-linear equations of evolution, with applications to partial differential equations, in: Lecture Notes in Math., vol. 448, Springer, 1975, pp. 25–70.

[4] F. Oliveira, Stability of the solitons for the one-dimensional Zakharov–Rubenchik equation, Physica D 175 (2003) 220–240.

[5] Y. Shibata, Y. Tsutsumi, Local existence of solutions for the initial boundary problem of fully nonlinear wave equation, Nonlinear Anal. TMA 11 (1987) 335–365.

[6] M. Tsutsumi, S. Hatano, Well-posedness of the Cauchy problem for the long wave–short wave resonance equations, Nonlinear Anal. TMA 22 (1994) 155–171.

[7] M. Tsutsumi, S. Hatano, Well-posedness of the Cauchy problem for Benney's first equations of long wave short wave interactions, Funkcial. Ekvac. 37 (1994) 289–316.