Available online at www.sciencedirect.com
ScienceDirect
C. R. Acad. Sci. Paris, Ser. I 344 (2007) 437–442
COMPTES RENDUS MATHEMATIQUE
http://france.elsevier.com/direct/CRASS1/

Partial Differential Equations

Calderón–Zygmund estimates for measure data problems

Giuseppe Mingione

Dipartimento di Matematica, Università di Parma, Viale G.P. Usberti 53/a, Campus, 43100 Parma, Italy

Received 17 November 2006; accepted 25 January 2007

Available online 13 March 2007

Presented by Alain Bensoussan

Abstract

New existence and regularity results are given for non-linear elliptic problems with measure data. The gradient of the solution is itself in an optimal (fractional) Sobolev space: this can be considered an extension of Calderón–Zygmund theory to measure data problems. ***To cite this article: G. Mingione, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

Estimations de type Calderon–Zygmund pour des problèmes avec données mesures. On établit de nouveaux résultats d'existence et régularité pour des problèmes elliptiques non-linéaires avec données mesures. Le gradient de la solution appartient lui-même à un espace de Sobolev (fractionnaire) optimal, ce que l'on peut considérer comme une extension de la théorie de Calderón–Zygmund aux problèmes avec données mesures. ***Pour citer cet article : G. Mingione, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Version française abrégée

Considérons le problème de Dirichlet (1) sous les hypothèses (2), (3), où μ est une mesure de Radon de variation totale finie. Un théorème classique de Boccardo et Gallouët [4,5] assure l'existence d'une solution, au sens des distributions, u de (1), dont le gradient Du vérifie les propriétés de régularité (5). Nous nous proposons ici de montrer qu'une telle propriété, unique résultat de régularité connu pour les solutions, est en fait une conséquence d'un résultat de régularité plus profond pour Du. Nous montrons en effet que Du appartient à un espace de Sobolev (fractionnaire) optimal : plus précisément, (9) a lieu. Nous obtenons alors en corollaire, le résulat d'intégrabilié (5), via le thèorème d'injection de Sobolev dans le cas fractionnaire. Nous obtenons également de substantielles améliorations des résultats, à la fois d'intégrabilité et de différentiabilité, dans le cas où la mesure μ vérifie la conditon de diffusion (15). Nous démontrons en effet, sous cette hypothèse, de nouvelles estimées optimales, de Marcinkiewicz pour le gradient, donnant ainsi une version non-linéaire d'un théorème de D. Adams [1]. La condition (15) permet également d'améliorer le taux de différentiabilité de Du ; nous prouvons en outre une estimée de décroissance de type Morrey pour

E-mail address: giuseppe.mingione@unipr.it.

1631-073X/$ – see front matter
doi:10.1016/j.crma.2007.02.005

la dérivée fractionnaire. Les démonstrations utilisent des arguments de localisation, reposant sur un certain type de décomposition dyadique du domaine, combinés avec des arguments de comparaison locale.

1. Introduction and function spaces

Let us consider the following Dirichlet problem:

$$-\operatorname{div} a(x, Du) = \mu \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{1}$$

Here we assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain, μ is a signed Radon measure with finite total variation $|\mu|(\Omega) < \infty$, and $a : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathèodory vector field satisfying the following standard monotonicity and Lipschitz assumptions:

$$\begin{cases} \nu\big(s^2 + |z_1|^2 + |z_2|^2\big)^{(p-2)/2}|z_2 - z_1|^2 \leqslant \big\langle a(x, z_2) - a(x, z_1), z_2 - z_1\big\rangle, \\ \big|a(x, z_2) - a(x, z_1)\big| \leqslant L\big(s^2 + |z_1|^2 + |z_2|^2\big)^{(p-2)/2}|z_2 - z_1|, \quad \big|a(x, 0)\big| \leqslant Ls^{p-1}, \end{cases} \tag{2}$$

for every $z_1, z_2 \in \mathbb{R}^n$, $x \in \Omega$. When referring to the structural properties of a, and in particular to (2), we shall always assume $p \geqslant 2,\ n \geqslant 2,\ 0 < \nu \leqslant L,\ s \geqslant 0$. The measure μ will be considered as defined on the whole $\mathbb{R}^n$ by simply letting $|\mu|(\mathbb{R}^n \setminus \Omega) = 0$. Sometimes, we shall also use the following assumption:

$$\big|a(x, z) - a(x_0, z)\big| \leqslant L|x - x_0|\big(s^2 + |z|^2\big)^{(p-1)/2}, \quad \text{for all } x, x_0 \in \Omega, \text{ and } z \in \mathbb{R}^n. \tag{3}$$

Assumptions (2) are modeled on the basic example $-\operatorname{div}[c(x)(s^2 + |Du|^2)^{\frac{p-2}{2}} Du] = \mu$, where $\nu \leqslant c(x) \leqslant L$. An important instance is the p-Laplacian operator with measure data: $-\Delta_p u = -\operatorname{div}(|Du|^{p-2}Du) = \mu$. For the problem (1) we shall adopt the following distributional-like notion of solution, compare with [4]:

Definition 1.1. A solution u to the problem (1) under assumptions (2), is a function $u \in W_0^{1,1}(\Omega)$ such that $a(x, Du) \in L^1(\Omega, \mathbb{R}^n)$ and

$$\int_\Omega a(x, Du) D\varphi \,\mathrm{d}x = \int_\Omega \varphi \,\mathrm{d}\mu \quad \text{for every } \varphi \in C_c^\infty(\Omega). \tag{4}$$

The study of problem (1) began with the fundamental work of Littman, Stampacchia and Weinberger [12,15], who treated linear equations with measurable coefficients. When referring to Definition 1.1, the existence theory for the general quasilinear Leray–Lions type operators in (1) has been established in the by now classical papers of Boccardo and Gallouët [4,5], who proved the existence of a solution u to (1) such that

$$Du \in L^q\big(\Omega, \mathbb{R}^n\big) \quad \text{for every } q < b := n(p-1)/(n-1) \text{ when } p \leqslant n. \tag{5}$$

The solution found in [4] is obtained via approximation with 'regular data', that is, solving auxiliary problems $-\operatorname{div} a(x, Du_n) = f_n$, $u_n \in W_0^{1,p}(\Omega)$, where f_n is a suitable mollification of μ; then a combination of a priori and convergence estimates yields the existence. These are the so-called Solutions Obtained as Limits of Approximations (SOLA), and are the solutions we are going to deal with here. Distributional solutions as (4) are not unique, and uniqueness is indeed the main open problem in the theory: identifying a suitable functional class where a unique solution can be defined and found. Despite several attempts such a problem remains mostly unsolved but when $p = n$ [9,10], or in the case where μ is an L^1-function [3,6,8] when so-called entropy and renormalized solutions come into the play.

Inclusion (5) is optimal in the scale of Lebesgue spaces, as $Du \notin L^b$ in general. The counterexample is given by the Green's function for the p-Laplacian operator, that is the (unique) solution to $\Delta_p u = \delta_0$, which is proportional to $|x|^{(p-n)/(n-1)}$; here δ_0 denotes the Dirac measure charging the origin. Anyway (5) can be sharpened using Marcinkiewicz spaces [3], see below, since

$$Du \in \mathcal{M}^b\big(\Omega, \mathbb{R}^n\big). \tag{6}$$

When $p > n$ instead, μ belongs to the dual of $W^{1,p}$, and the existence of a unique solution in the natural space $W_0^{1,p}(\Omega)$ follows by standard methods of the theory of monotone operators.

To explain our results, that will be presented in the next section, we shall need the following preliminary definitions. For a bounded open set $A \subset \mathbb{R}^n$ and $k \in \mathbb{N}$, parameters $\alpha \in (0,1)$ and $q \in [1,\infty)$, a map $w : A \to \mathbb{R}^k$ belongs to the fractional Sobolev space $W^{\alpha,q}(A,\mathbb{R}^k)$, see [2], provided:

$$\|w\|^q_{W^{\alpha,q}(A)} := \int_A \big|w(x)\big|^q \,\mathrm{d}x + \int_A\int_A \frac{|w(x)-w(y)|^q}{|x-y|^{n+\alpha q}}\,\mathrm{d}x\,\mathrm{d}y =: \|w\|^q_{L^q(A)} + [w]^q_{\alpha,q;A} < \infty.$$

A measurable map $w : A \to \mathbb{R}^k$ belongs to the Marcinkiewicz space $\mathcal{M}^t(A,\mathbb{R}^k)$ iff

$$\sup_{\lambda \geqslant 0} \lambda^t \big|\big\{x \in A\colon\ |w| > \lambda\big\}\big| =: \|w\|^t_{\mathcal{M}^t(A)} < \infty.$$

Now Marcinkiewicz–Morrey spaces [1]; a map $w \in \mathcal{M}^t(A,\mathbb{R}^k)$ belongs to the space $\mathcal{M}^{t,\theta}(A,\mathbb{R}^k)$ with $\theta \in [0,n]$ iff, with B_R denoting a generic ball of radius R,

$$\|w\|^t_{\mathcal{M}^{t,\theta}(A)} := \|w\|^t_{\mathcal{M}^t(A)} + \sup_{B_R \subset A,\, R \leqslant 1} R^{\theta-n}\|w\|^t_{\mathcal{M}^t(B_R)} < \infty. \tag{7}$$

Obviously $\|w\|_{\mathcal{M}^{t,n}(A)} \approx \|w\|_{\mathcal{M}^t(A)}$, and $\|w\|_{\mathcal{M}^{t,\theta_1}(A)} < \|w\|_{\mathcal{M}^{t,\theta_2}(A)}$ iff $\theta_1 < \theta_2$. Beside, we recall the definition of Sobolev–Morrey spaces of fractional order. We say that a map $w \in W^{\alpha,q}(A,\mathbb{R}^k)$, belongs to $W^{\alpha,q,\theta}(A,\mathbb{R}^k)$, with $\alpha \in (0,1)$, $q \in [1,\infty)$, $\theta \in [0,n]$, iff $w \in W^{\alpha,q}(A,\mathbb{R}^k)$, and moreover,

$$[w]^q_{\alpha,q,\theta;A} := \sup_{B_R \subset A,\, R \leqslant 1} R^{\theta-n}[w]^q_{\alpha,q;B_R} < \infty. \tag{8}$$

In any case we let $\|w\|_{W^{\alpha,q,\theta}(A)} := \|w\|_{W^{\alpha,q}(A)} + [w]_{\alpha,q,\theta;A}$. See [7,13], and enclosed references. The local variants of all the above spaces are then defined in the usual way.

2. Calderón–Zygmund theory

Here we describe some of the main results in [14]. Amongst the possible definitions of solutions we take the one of SOLA for simplicity; other notions, as entropy solutions, maybe also used: the main emphasis here is on a priori *regularity estimates*. Inclusion (5), together with its refinement (6), is basically the only regularity result available for non-linear measure data problems as (1): we shall see that it is actually a consequence of a much deeper regularity property of Du. Let us focus on the case $p = 2$, considering $\Delta u = f$; *the standard Calderón–Zygmund theory* now asserts that $f \in L^{1+\varepsilon}$ implies $Du \in W^{1,1+\varepsilon}$ if $\varepsilon > 0$.

By Sobolev's embedding theorem, in particular $Du \in L^{n/(n-1)}$, that is the limit case of (5). This fails for $\varepsilon = 0$, and so does the inclusion $Du \in W^{1,1}$ in general. So one could interpret (5) *as the trace of a potentially existent Calderón–Zygmund theory below the limit case* $W^{1,1}$. Indeed the following holds:

Theorem 2.1 *(of Calderón–Zygmund type). Under the assumptions* (2) *and* (3) *with* $2 \leqslant p \leqslant n$*, there exists a solution* $u \in W^{1,1}_0(\Omega)$ *to the problem* (1) *such that for every* $\varepsilon \in (0,1)$*,*

$$Du \in W^{\frac{1-\varepsilon}{p-1},p-1}_{\mathrm{loc}}\big(\Omega,\mathbb{R}^n\big), \quad \textit{and in particular} \quad Du \in W^{1-\varepsilon,1}_{\mathrm{loc}}\big(\Omega,\mathbb{R}^n\big) \quad \textit{when } p = 2. \tag{9}$$

More in general, with $b = n(p-1)/(n-1)$ *as in* (5)*, for every* $\varepsilon \in (0,\sigma(q))$ *it holds*:

$$Du \in W^{\frac{\sigma(q)-\varepsilon}{q},q}_{\mathrm{loc}}\big(\Omega,\mathbb{R}^n\big) \quad \textit{for every } q \in [p-1,b),\ \textit{where } \sigma(q) := n(1-q/b). \tag{10}$$

Furthermore there exists a constant $c \equiv c(n,p,L/\nu,q,\sigma)$ *such that for every ball* $B_R \Subset \Omega$ *of radius* $R > 0$*, and* $\sigma \in (0,\sigma(q))$*, the following naturally scaling local estimate holds*:

$$\int_{B_{R/2}}\int_{B_{R/2}} \frac{|Du(x)-Du(y)|^q}{|x-y|^{n+\sigma}}\,\mathrm{d}x\,\mathrm{d}y \leqslant \frac{c}{R^\sigma}\int_{B_R}\big(|Du|^q + s^q\big)\,\mathrm{d}x + cR^{\sigma(q)-\sigma}\big[|\mu|(\overline{B_R})\big]^{q/(p-1)}. \tag{11}$$

In other words, when $p = 2$ though in general $Du \notin W^{1,1}$, by (9) we 'almost have' *second derivatives of* u, and in this sense the previous theorem extends the Calderón–Zygmund theory to problems with measure data. Observe that (9) is sharp for every $p \geqslant 2$; indeed assuming by contradiction that $Du \in W^{1/(p-1),p-1}$, using Sobolev embedding theorem in the fractional case [2] would give exactly $Du \in L^b$, but this is impossible as seen in the previous section. A similar argument gives the sharpness of (10), which in turn is obtainable from (9) also via interpolation. On the other hand, (9) locally implies (5), again via Sobolev embedding. When $p \neq 2$ we do not approach the differentiability exponent 1 in (9) as $\varepsilon \searrow 0$, but only $1/(p-1)$. This is not a surprise: even for the model case $\Delta_p u = 0$ the existence of second derivatives of $W^{1,p}$-solutions is not clear due to the degeneracy of the problem, while fractional derivatives naturally appear: $Du \in W^{2/p,p}$. On the other hand, a classical result going back to K. Uhlenbeck asserts that though Du may be not differentiable for $\Delta_p u = 0$, *certain natural nonlinear expressions of the gradient still are* (in T. Iwaniec's words). Indeed, defining $V(Du) := (s^2 + |Du|^2)^{(p-2)/4} Du$, under assumptions (2) it turns out that $V(Du) \in W^{1,2}_{\mathrm{loc}}(\Omega)$ for any $W^{1,p}$-solution to $\operatorname{div} a(Du) = 0$. This phenomenon extends to measure data problems:

Theorem 2.2 *(Non-linear Calderón–Zygmund estimate). Under the assumptions* (2) *and* (3) *with* $2 \leqslant p \leqslant n$, *let* $u \in W^{1,q}_0(\Omega)$ *be the solution to* (1) *found in Theorem* 2.1. *Then*

$$V(Du) \in W^{1-\varepsilon,q_0}_{\mathrm{loc}}\big(\Omega, \mathbb{R}^n\big), \quad \textit{for every } \varepsilon \in (0,1),\ q_0 := 2p/(p+2). \tag{12}$$

Moreover, for any open subset $\Omega' \Subset \Omega$, *we have, with* $\tilde{c} \equiv \tilde{c}(n, p, L/\nu, \varepsilon, \operatorname{dist}(\Omega', \partial\Omega), \Omega)$,

$$\big[V(Du)\big]_{1-\varepsilon,q_0;\Omega'} \leqslant \tilde{c}\big[|\mu|(\Omega)\big]^{p/(2p-2)} + \tilde{c} s^{p/2} |\Omega|^{1/q_0}. \tag{13}$$

When the problem is non-degenerate, that is $s > 0$ in (2), 'almost second derivatives' are again retrievable:

Theorem 2.3 *(The non-degenerate case). Under the assumptions* (2) *and* (3) *with* $2 \leqslant p \leqslant n$, *let* $u \in W^{1,q}_0(\Omega)$ *be the solution to* (1) *found in Theorem* 2.1, *and assume* $s > 0$. *Then*

$$Du \in W^{1-\varepsilon,q_0}_{\mathrm{loc}}\big(\Omega, \mathbb{R}^n\big), \quad \textit{for every } \varepsilon \in (0,1),\ q_0 := 2p/(p+2).$$

Estimate (13) *holds with* Du *replacing* $V(Du)$ *where the constant* $\tilde{c}$ *is replaced by* $s^{(2-p)/2} c(n,p)\tilde{c}$.

When $p > 2$, we get different differentiability rates too:

Theorem 2.4 *(A different range). Under the assumptions* (2), *and* (3) *with* $2 \leqslant p \leqslant n$, *let* $u \in W^{1,q}_0(\Omega)$ *be the solution to* (1) *found in Theorem* 2.1. *Then*

$$Du \in W^{2/p-\varepsilon, pq_0/2}_{\mathrm{loc}}\big(\Omega, \mathbb{R}^n\big) \quad \textit{for every } \varepsilon \in (0,1),\ q_0 := 2p/(p+2). \tag{14}$$

3. Diffusing measures

The sharpness of (6) and (9)–(10) stems from considering counterexamples involving Dirac measures, see Section 1. How things change when considering measures diffusing on sets with higher Hausdorff dimension? A natural way to quantify this is to consider the following Morrey-type density condition:

$$|\mu|(B_R) \leqslant M R^{n-\theta}, \quad 0 \leqslant \theta \leqslant n,\ M \geqslant 0, \tag{15}$$

satisfied for any ball $B_R \subset \mathbb{R}^n$ of radius R. Assuming (15) does not allow μ to concentrate on sets with Hausdorff dimension less than $n - \theta$, and indeed higher regularity of solutions follows. We shall restrict to the case $\theta \in [p, n]$, otherwise the measure μ belongs to the dual space $W^{-1,p'}$ by a fundamental theorem of D. Adams, and the existence of a unique solution in $W^{1,p}_0(\Omega)$ for (1) follows via monotonicity methods; see [14] for results in this case. Assuming (15) we discover that in all the above statements *the role of the dimension* n *is actually played by* θ. For the first improvement regards the integrability of Du, simultaneously detectable in two different scales, the Marcinkiewicz and Morrey ones, see (7).

Theorem 3.1 *(Marcinkiewicz–Morrey regularity). Under the assumptions* (2) *with* $2 \leqslant p \leqslant n$, *and* (15) *with* $\theta \geqslant p$, *there exists a solution* $u \in W^{1,1}_0(\Omega)$ *to the problem* (1) *such that*

$$Du \in \mathcal{M}^{m,\theta}_{\mathrm{loc}}\big(\Omega, \mathbb{R}^n\big) \subseteq \mathcal{M}^{m}_{\mathrm{loc}}\big(\Omega, \mathbb{R}^n\big), \quad m := \theta(p-1)/(\theta-1). \tag{16}$$

Moreover, for any open subset $\Omega' \Subset \Omega$ *we have*

$$\|Du\|_{\mathcal{M}^{m,\theta}(\Omega')} \leqslant cM^{1/(p-1)} + cs|\Omega|^{1/m},$$

where $c \equiv c(n, p, L/\nu, \operatorname{dist}(\Omega', \partial\Omega), \Omega)$*, and* M *is in* (15). *In particular, in the limit case* $\theta = p$ *we have*

$$Du \in \mathcal{M}^{p,p}_{\mathrm{loc}}\big(\Omega, \mathbb{R}^n\big) \subseteq \mathcal{M}^{p}_{\mathrm{loc}}\big(\Omega, \mathbb{R}^n\big). \tag{17}$$

The exponent m in (16) is expected to be the best possible for every $p \geqslant 2$, and it actually is when $p = 2$: in general $Du \notin L^{\theta/(\theta-1)}$, even locally. Indeed, Theorem 3.1 may as well be considered as the non-linear version of a classical potential theory result of D. Adams [1], asserting that $I_\alpha \in \mathcal{M}^{\theta/(\theta-\alpha),\theta}$ whenever μ satisfies (15), where $I(\mu)(x) := \int |x-y|^{\alpha-n}\,\mathrm{d}\mu(y)$, $\alpha \in (0,\theta)$; this applies to $\Delta u = \mu$, when $Du \approx I_1(\mu)$. Adams's result is sharp, hence the sharpness of (16) for $p = 2$; see [14] for more. Observe that for $\theta < n$ it is $\mathcal{M}^{m,\theta} \subset \mathcal{M}^{m,n} \equiv \mathcal{M}^m$, therefore $Du \in \mathcal{M}^m$ as in (16)–(17), a result that naturally improves (6). As explained above, when $\theta < p$, the solution u is uniquely found in $W^{1,p}_0(\Omega)$, so that (15) provides the natural scale allowing to pass from (6), when $\theta = n$, to (17), when $\theta = p$; in this last case the $W^{1,p}$-regularity of the solution is missed just by a natural Marcinkiewicz-scale factor. In other words (17) is the borderline result before passing to the case of capacitary measures. Finally, note that (16) does not require (3). Note how in Theorem 3.1 assumption (3) is not needed.

The second effect of condition (15) is an expansion of the differentiability range in (10). Moreover the fractional derivatives are themselves in a Morrey space, see the definition in (8).

Theorem 3.2 *(Sobolev–Morrey regularity). Under the assumptions* (2) *and* (3) *with* $p \leqslant n$*, and* (15) *with* $\theta \geqslant p$*, let* $u \in W^{1,1}_0(\Omega)$ *be the solution found in Theorem* 3.1. *Then*

$$Du \in W^{\frac{\sigma(q,\theta)-\varepsilon}{q},q,\theta}_{\mathrm{loc}}\big(\Omega, \mathbb{R}^n\big) \quad \text{for every } q \in [p-1, m), \text{ where } \sigma(q) := \theta(1 - q/m). \tag{18}$$

In particular,

$$Du \in W^{\frac{1-\varepsilon}{p-1},p-1,\theta}_{\mathrm{loc}}\big(\Omega, \mathbb{R}^n\big), \quad \text{and} \quad Du \in W^{1-\varepsilon,1,\theta}_{\mathrm{loc}}\big(\Omega, \mathbb{R}^n\big) \text{ when } p = 2. \tag{19}$$

Moreover, for any open subset $\Omega' \Subset \Omega$ *and* $\sigma \in (0, \sigma(q,\theta))$*, there exists* $c \equiv c(n, p, L/\nu, q, \sigma, \operatorname{dist}(\Omega', \partial\Omega), \Omega)$ *such that*

$$\|Du\|_{W^{\sigma/q,q,\theta}(\Omega')} \leqslant cM^{1/(p-1)} + cs|\Omega|^{1/q}.$$

The latter estimate extends to the case of non-linear equations with measure data the classical Morrey space estimates for linear elliptic equations, see for instance [11]: the standard result for the model case $\Delta u = f$ is that $Du \in W^{1,q,\theta}$ when $f \in L^{q,\theta}$ for $q > 1$; inclusion (19) is a sharp extension of this to the case $q = 1$, and we refer again to [14] for more results. In light of (16) we can interpret (18) and therefore also (10) as a scale of regularity for Du that leads, as $q \nearrow m$, from the maximal differentiability (19) toward the maximal integrability (16).

The proofs of all the above theorems are contained in [14], and they are quite technical. As for the differentiability estimates, a careful localization procedure is used: we employ a decomposition of the domain in cubes C with mesh h. On everyone of them we solve a homogeneous problem of the type $\operatorname{div} a(x, Dv) = 0$ in C, and $v \equiv u$ on ∂C; for Dv good differentiability estimates are available. Then we prove a careful integral estimate on the difference $|Du - Dv|$, in terms of the mesh-size h. Finally we conclude with a covering-comparison argument, and a delicate iteration scheme in fractional Sobolev spaces. As for the new Marcinkiewicz estimate of Theorem 3.1, we shall again use a localization argument, but this time on certain suitable Calderón–Zygmund cubes obtained decomposing the level set of the gradient of the solution. Once again we shall proceed via a comparison technique, but a crucial point will be here proving the a priori Morrey-space regularity of the gradient Du; this gives in turn a better rate of decay of the level set measure, on each one of the Calderón–Zygmund cubes considered.

References

[1] D.R. Adams, A note on Riesz potentials, Duke Math. J. 42 (1975) 765–778.
[2] R.A. Adams, Sobolev Spaces, Academic Press, New York, 1975.

[3] P. Benilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, J.L. Vázquez, An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 22 (1995) 241–273.

[4] L. Boccardo, T. Gallouët, Nonlinear elliptic and parabolic equations involving measure data, J. Funct. Anal. 87 (1989) 149–169.

[5] L. Boccardo, T. Gallouët, Nonlinear elliptic equations with right-hand side measures, Comm. Partial Differential Equations 17 (1992) 641–655.

[6] L. Boccardo, T. Gallouët, L. Orsina, Existence and uniqueness of entropy solutions for nonlinear elliptic equations with measure data, Ann. Inst. H. Poincaré Anal. Non Linéaire 13 (1996) 539–551.

[7] S. Campanato, Proprietà di una famiglia di spazi funzionali, Ann. Scuola Norm. Sup. Pisa (3) 18 (1964) 137–160.

[8] G. Dal Maso, F. Murat, L. Orsina, A. Prignet, Renormalized solutions of elliptic equations with general measure data, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 28 (1999) 741–808.

[9] G. Dolzmann, N. Hungerbühler, S. Müller, Uniqueness and maximal regularity for nonlinear elliptic systems of n-Laplace type with measure valued right hand side, J. Reine Angew. Math. (Crelles J.) 520 (2000) 1–35.

[10] L. Greco, T. Iwaniec, C. Sbordone, Inverting the p-harmonic operator, Manuscripta Math. 92 (1997) 249–258.

[11] G.M. Lieberman, A mostly elementary proof of Morrey space estimates for elliptic and parabolic equations with VMO coefficients, J. Funct. Anal. 201 (2003) 457–479.

[12] W. Littman, G. Stampacchia, H.F. Weinberger, Regular points for elliptic equations with discontinuous coefficients, Ann. Scuola Norm. Sup. Pisa (3) 17 (1963) 43–77.

[13] A.L. Mazzucato, Besov–Morrey spaces: function space theory and applications to non-linear PDE, Trans. Amer. Math. Soc. 355 (2003) 1297–1364.

[14] G. Mingione, The Calderón–Zygmund theory for elliptic problems with measure data, Preprint, September 2006; available at http://www.unipr.it/~mingiu36.

[15] G. Stampacchia, Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus, Ann. Inst. Fourier 15 (1965) 189–258.