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Partial Differential Equations

An existence theorem for a 2-D coupled sedimentation shallow-water model

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Abstract

We present an existence theorem of a two-dimensional sedimentation model coupling a shallow-water system with a sediment transport equation. A finite dimensional problem is solved using a Brouwer fix point theorem. We prove that the limits of the resulting solution sequences satisfy the model equations. ***To cite this article: B. Toumbou et al., C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

Un théorème d'existence pour un modèle couplé 2D de Saint-Venant et de sédimentation. Nous présentons un théorème d'existence d'un modèle bidimensionnel de sédimentation composé d'un système de Saint-Venant et d'une équation de transport de sédiment. Nous résolvons un problème de dimension finie utilisant un théorème de point fixe de Brouwer. Nous montrons que les limites des suites de solutions de ce problème de dimension finie satisfont les équations du modèle. ***Pour citer cet article : B. Toumbou et al., C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

1. Introduction

A number of theoretical results have been obtained for shallow water models. Indeed, in [5] an existence theorem for a shallow-water model including a rotational term is obtained. In [1] the existence of global weak solutions of a viscous shallow-water model with friction term is demonstrated. In [2], the sediment transport part is explored numerically and a discontinuous Galerkin method is developed. Numerical methods for coupled problems are also developed in [4,3].

However, the coupling between shallow-water and sediment transport models is a research area where there is a lack of theoretical results. In this study, we couple a shallow-water model with a sediment transport equation. The shallow-water part is obtained in (1)–(4) by integrating the 3D Navier–Stokes equations over the fluid layer by taking into account the bed evolution,

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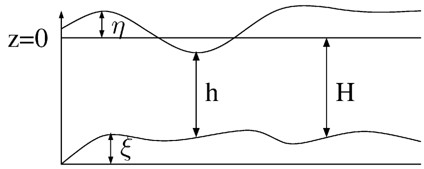

Fig. 1. The model under study.

$$h_t + \nabla\cdot(h\mathbf{u}) = 0 \quad \text{in } Q = \Omega\times]0,T[, \tag{1}$$

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} + g\nabla(h-H) - \nu\triangle\mathbf{u} = f \quad \text{in } Q, \tag{2}$$

$$\mathbf{u} = 0 \quad \text{in } \partial\Omega\times]0,T[, \tag{3}$$

$$h(0) = h_0 \quad \text{in } \Omega, \quad \mathbf{u}(0) = \mathbf{u}_0 \quad \text{in } \Omega, \tag{4}$$

where $h(\mathbf{x},t)$ is the height of the water column, $H(\mathbf{x},t)$ is the evolution of the bottom and $\xi(\mathbf{x},t)$ is the thickness of the sediment layer, as shown in Fig. 1 with $\eta = h - H$. The flow velocity is $\mathbf{u} = (u, v)$, f is the external resulting force, ν is the viscosity coefficient, g is the gravitational acceleration and $T > 0$ is a real number.

For the sediment transport equation we have the following general model given in [2]

$$\xi_t + \nabla\cdot\big(A|\mathbf{u}|^{n-1}\mathbf{u}\big) = 0 \quad \text{in } Q. \tag{5}$$

In [2], Eq. (5) is solved numerically without theoretical analysis for $n = 1$ and A constant (height). Here, we choose $n = 1$ and $A = h(\mathbf{x},t)$. However, assuming that $\eta \ll H$ and $\nabla\eta \ll \nabla H$, we replace $h = H + \eta$ by H and then $A = H(\mathbf{x},t)$. Such assumptions are largely used to model lakes and oceans and they are chosen here for the sake of simplicity and for subsequent applications in Guiers Lake (Senegal). Since $\xi + H$ is constant in time (and in space), then $\xi_t = -H_t$ and Eq. (5) leads to

$$-H_t + \nabla\cdot(H\mathbf{u}) = 0 \quad \text{in } Q, \quad \text{with } H(0) = H_0 \quad \text{in } \Omega. \tag{6}$$

We take $\mathbf{u}_0 \in (H_0^1(\Omega))^2$, $(h_0, H_0) \in (L^1(\Omega))^2$, $h_0 \geqslant 0$, $H_0 \geqslant 0$ and $f \in L^2(0,T;(H^{-1}(\Omega))^2)$.

The Note is organized as follows. In Section 2 we give some preliminary estimations. In Section 3 we state and prove the existence theorem for the coupled model (1)–(6).

2. Preliminary estimations

Let $V = (H_0^1(\Omega))^2$ and denote by $(\cdot,\cdot)$ the scalar product of $L^2(\Omega)$ or $(L^2(\Omega))^2$ and $\|\cdot\|$ its associated norm. Further, $(\cdot,\cdot)_{-1,1}$ denotes the duality product between $(H^{-1}(\Omega))^2$ and $(H_0^1(\Omega))^2$. Let $\{\mathbf{v}_1,\ldots,\mathbf{v}_n,\ldots\}$ be an Hilbertian basis of V, $\mathbf{v}_n \in (H^m(\Omega))^2$, $m \geqslant 3$ and $V_n = \operatorname{span}\{\mathbf{v}_1,\ldots,\mathbf{v}_n\}$. Then for $\mathbf{u}_n(t) \in V_n$, we have $\mathbf{u}_n(t) = \sum_{i=1,\ldots,n} a_i(t)\mathbf{v}_i$ and from (1)–(6) we obtain the following finite dimensional problem

$$h_{n,t} + \nabla\cdot(h_n\mathbf{u}_n) = 0 \quad \text{in } Q, \tag{7}$$

$$\mathbf{u}_{n,t} + (\mathbf{u}_n\cdot\nabla)\mathbf{u}_n + g\nabla(h_n - H_n) - \nu\triangle\mathbf{u}_n = f \quad \text{in } Q, \tag{8}$$

$$H_{n,t} - \nabla\cdot(H_n\mathbf{u}_n) = 0 \quad \text{in } Q, \tag{9}$$

$$\mathbf{u}_n = 0 \quad \text{in } \partial\Omega\times]0,T[, \quad h_n(t=0) = h_{n,0} \geqslant 0, \quad H_n(t=0) = H_{n,0} \geqslant 0, \quad \mathbf{u}_n(t=0) = \mathbf{u}_{n,0} \quad \text{in } \Omega. \tag{10}$$

Multiplying (8) by $\mathbf{v} \in V_n$ leads to the following variational problem

$$(\mathbf{u}_{n,t},\mathbf{v}) + \big((\mathbf{u}_n\cdot\nabla)\mathbf{u}_n,\mathbf{v}\big) + \big(g\nabla(h_n - H_n),\mathbf{v}\big) - \nu\big(\nabla\cdot(\nabla\mathbf{u}_n),\mathbf{v}\big) = (f,\mathbf{v}), \quad \forall\mathbf{v}\in V_n. \tag{11}$$

Replacing $\mathbf{v}$ by $\mathbf{u}_n(t)$ in (11) and using (7) and (9) we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{u}_n,\mathbf{u}_n) + g\frac{\mathrm{d}}{\mathrm{d}t}(h_n\log h_n - h_n, 1) + g\frac{\mathrm{d}}{\mathrm{d}t}(H_n\log H_n - H_n, 1) + \nu\|\mathbf{u}_n\|_V^2$$
$$\leqslant \frac{1}{2\lambda}\|f\|^2_{(H^{-1}(\Omega))^2} + \frac{\lambda}{2}\|\mathbf{u}_n\|_V^2 + \frac{C}{2}\|\mathbf{u}_n\|_V^2\|\mathbf{u}_n\|, \quad \forall\lambda > 0, \text{ with } C = 3(C_1 + C_2). \tag{12}$$

The constants C_1, C_2 and λ verify the Young and Gagliardo–Nirenberg inequalities

$$\|u_n\|^2_{L^4(\Omega)} \leqslant C_1\|u_n\|_{H^1_0(\Omega)}\|u_n\|; \qquad \|v_n\|^2_{L^4(\Omega)} \leqslant C_2\|v_n\|_{H^1_0(\Omega)}\|v_n\|;$$
$$\|f\|_{H^{-1}(\Omega)^2}\|\mathbf{u}_n\|_V \leqslant \frac{1}{2\lambda}\|f\|^2_{H^{-1}(\Omega)^2} + \frac{\lambda}{2}\|\mathbf{u}_n\|^2_V.$$

Setting $D = 2\nu - \lambda$ and integrating (12) between 0 and $t \in]0, T[$, we obtain

$$\big\|\mathbf{u}_n(t)\big\|^2 + 2g\int_\Omega h_n(t)\log h_n(t)\,\mathrm{d}\mathbf{x} + 2g\int_\Omega H_n(t)\log H_n(t)\,\mathrm{d}\mathbf{x} + \int_0^t \big(D - C\big\|\mathbf{u}_n(s)\big\|\big)\big\|\mathbf{u}_n(s)\big\|^2_V\,\mathrm{d}s$$
$$\leqslant \frac{1}{\lambda}\int_0^t \big\|f(s)\big\|^2_{(H^{-1}(\Omega))^2}\,\mathrm{d}s + \|\mathbf{u}_{n,0}\|^2 + 2g\int_\Omega h_{n,0}\log h_{n,0}\,\mathrm{d}\mathbf{x} + 2g\int_\Omega H_{n,0}\log H_{n,0}\,\mathrm{d}\mathbf{x}. \tag{13}$$

3. Existence theorem

Theorem 3.1. *Let $\mathbf{u}_0 \in V$, $(h_0, H_0) \in (L^1(\Omega))^2$, $h_0 \geqslant 0$, $H_0 \geqslant 0$, and f verify*

$$\frac{1}{\lambda}\|f\|^2_{L^2(0,T;H^{-1}(\Omega)^2)} + \|\mathbf{u}_0\|^2 + 2g\int_\Omega h_0\log h_0\,\mathrm{d}\mathbf{x} + 2g\int_\Omega H_0\log H_0\,\mathrm{d}\mathbf{x} + \frac{4g}{e}\,\mathrm{meas}(\Omega) < \frac{D^2}{C^2}, \tag{14}$$

then there exists $\mathbf{u} \in L^2(0,T,V) \cap L^\infty(0,T;(L^2(\Omega))^2)$, $h \in L^\infty(0,T;L^1(\Omega))$ and $H \in L^\infty(0,T;L^1(\Omega))$ verifying (1)–(6) *where* $\mathrm{meas}(\Omega)$ *denotes the measure of Ω.*

The proof of this theorem is given at the end of this section, after Lemmas 3.2, 3.3 and 3.4 are established. Small enough data are necessary to have $D - C\|\mathbf{u}_n\|_{L^\infty(0,T;(L^2(\Omega))^2)} > 0$. This inequality is obtained by using (14) and the continuity of $\mathbf{u}_n$ on $[0, T]$, and leads to

$$\big\|\mathbf{u}_n(t)\big\|^2 + 2g\int_\Omega \big(h_n(t)\log h_n(t) - h_{n,0}\log h_{n,0}\big)\,\mathrm{d}\mathbf{x} + 2g\int_\Omega \big(H_n(t)\log H_n(t) - H_{n,0}\log H_{n,0}\big)\,\mathrm{d}\mathbf{x}$$
$$\leqslant \frac{1}{\lambda}\|f\|^2_{L^2(0,T;(H^{-1}(\Omega))^2)} + \|\mathbf{u}_{n,0}\|^2 - \big(D - C\|\mathbf{u}_n\|_{L^\infty(0,T;(L^2(\Omega))^2)}\big)\int_0^t \big\|\mathbf{u}_n(s)\big\|^2_V\,\mathrm{d}s. \tag{15}$$

We now introduce the following lemmas in order to demonstrate Theorem 3.1.

Lemma 3.2. *Let $\mathbf{u}_n$, h_n and H_n be three sequences such that*

$$h_{n,t} + \nabla\cdot(\mathbf{u}_n h_n) = 0, \quad h_n \to h \text{ in } L^2\big(0,T;L^1(\Omega)\big) \text{ weakly}, \qquad h_{n,0} \to h_0 \quad \text{in } L^1(\Omega), \tag{16}$$
$$H_{n,t} - \nabla\cdot(\mathbf{u}_n H_n) = 0, \quad H_n \to H \text{ in } L^2\big(0,T;L^1(\Omega)\big) \text{ weakly}, \qquad H_{n,0} \to H_0 \quad \text{in } L^1(\Omega), \tag{17}$$
$$\mathbf{u}_n \in L^2\big(0,T;\big(H^m(\Omega)\big)^2\big), \quad m \geqslant 3, \quad h_n \in L^\infty\big(0,T;L^1(\Omega)\big), \quad \mathbf{u}_n h_n \in L^2\big(0,T;\big(L^1(\Omega)\big)^2\big), \tag{18}$$
$$\mathbf{u}_n \to \mathbf{u} \quad \text{in } L^2(0,T;V) \text{ weakly}, \qquad \mathbf{u}_{n,0} \to \mathbf{u}_0 \quad \text{in } V, \tag{19}$$
$$\int_Q h_n\theta\,\mathrm{d}x\,\mathrm{d}t \to \int_Q h\theta\,\mathrm{d}x\,\mathrm{d}t, \quad \int_Q H_n\theta\,\mathrm{d}x\,\mathrm{d}t \to \int_Q H\theta\,\mathrm{d}x\,\mathrm{d}t \quad \forall\theta \in L^1\big(0,T;L^\infty(\Omega)\big), \tag{20}$$

then we have

$$\mathbf{u}_n h_n \to \mathbf{u}h \quad \text{in } \big(L^1(Q)\big)^2 \text{ weakly}, \qquad \mathbf{u}_n H_n \to \mathbf{u}H \quad \text{in } \big(L^1(Q)\big)^2 \text{ weakly}. \tag{21}$$

Lemma 3.3. *Let* $(\mathbf{u}_n, h_n, H_n)$ *be solution of* (11) *such that* $h_n \to h$ *in* $L^2(0,T;L^1(\Omega))$ *weakly,* $H_n \to H$ *in* $L^2(0,T;L^1(\Omega))$ *weakly. Then* $\mathbf{u}_n$ *verifies* (13) *and we can extract from* $\mathbf{u}_n$ *a subsequence, denoted also by* $\mathbf{u}_n$*, such that*

$$\mathbf{u}_n \to \mathbf{u} \quad \text{in } L^2(0,T;V) \text{ weakly,} \tag{22}$$

$$\mathbf{u}_n \to \mathbf{u} \quad \text{in } L^\infty\big(0,T;\big(L^2(\Omega)\big)^2\big) \text{ weak star,} \tag{23}$$

$$(\mathbf{u}_n \cdot \nabla)\mathbf{u}_n \to (\mathbf{u} \cdot \nabla)\mathbf{u} \quad \text{in } L^{4/3}\big(0,T;\big(L^{4/3}(\Omega)\big)^2\big) \text{ weakly,} \tag{24}$$

$$\mathbf{u}_{n,t} \text{ is bounded in } L^{4/3}\big(0,T;\big(H^{-3}(\Omega)\big)^2\big), \tag{25}$$

and the limit $\mathbf{u}$ *of* $\mathbf{u}_n$ *verifies*

$$\frac{1}{2}(\mathbf{u}_t, \mathbf{v}) + \big((\mathbf{u}\cdot\nabla)\mathbf{u}, \mathbf{v}\big) - g(h, \nabla\cdot\mathbf{v}) + g(H, \nabla\cdot\mathbf{v}) + \nu(\nabla\mathbf{u}, \nabla\mathbf{v}) = (f, \mathbf{v})_{-1,1} \quad \forall \mathbf{v} \in \big(H^3(\Omega)\big)^2 \cap V. \tag{26}$$

Lemma 3.4. *If* $\mathbf{u}_{n,0} \in V_n \cap (H^m(\Omega))^2$*,* $m \geqslant 3$*,* $(h_{n,0}, H_{n,0}) \in (C^1(\overline{\Omega}))^2$*, then* (7)–(10) *has a solution* $(\mathbf{u}_n, h_n, H_n) \in L^2(0,T;V_n) \cap L^\infty(0,T;(L^2(\Omega))^2) \times (C^1(\overline{Q}))^2$.

Proof of Lemma 3.4. We use the Brouwer fixed point theorem. Indeed, we replace $\mathbf{u}_n$ in (7), (9) and (10) by a fix $\mathbf{w} \in L^2(0,T;V_n)$ and (7) and (9), with their respective initial conditions, are solved using the Galerkin method which leads to the solutions k and l, respectively. Then, we replace h_n and H_n by k and l in (8), respectively, and we solve the resulting problem to obtain $\mathbf{u}_n$. We verify that the application

$$\pi : \begin{pmatrix} B'(0,R) \subset L^2(0,T;V_n) \to B'(0,R) \subset L^2(0,T;V_n) \\ \mathbf{w} \mapsto \mathbf{u}_n \end{pmatrix}$$

meets the Brouwer fix point theorem conditions, where $B'(0,R)$ is the closed ball of radius R. Indeed, we use the weak topology of $L^2(0,T;V_n)$ in which $B'(0,R)$ is compact. Finally, we show that the application π is continuous. □

Positiveness of h_n. Since $\mathbf{u}_n \in C^0([0,T];(C^1(\overline{\Omega}))^2)$ then, from (7), we prove that $h_n \geqslant 0$. The proof of $H_n \geqslant 0$ is done in the same way by using (9).

Proof of Theorem 3.1. Relations (21) and (26) give (1), (6) and (2), respectively. To end the proof we need to show that $(\mathbf{u}, h, H)$ verifies (3), (4) and (6). Using (1) and (21) we obtain $h_t = -\nabla\cdot(\mathbf{u}h) \in L^2(0,T;W^{-1,1}(\Omega))$, then $h \in W^{1,2}(0,T;W^{-1,1}(\Omega))$. Since the embedding $W^{1,p}(0,T) \subset C([0,T])$ is compact if $1 < p \leqslant \infty$, and thanks to (16), we have $h(t=0) = h_0$. By using (2) we have $\mathbf{u}_t \in L^{4/3}(0,T;(H^{-3}(\Omega))^2)$ and then (22) gives $\mathbf{u} \in W^{1,4/3}(0,T;(H^{-3}(\Omega))^2)$. This implies that $\mathbf{u}$ is continuous in $[0,T]$ and thanks to (19), we have $\mathbf{u}(t=0) = \mathbf{u}_0$ and $\mathbf{u} = 0$ in $\partial\Omega\times]0,T[$. □

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